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OFF-SHELL SUPERSPACE $D = 10$ SUPER-YANG-MILLS FROM A COVARIANTLY QUANTIZED GREEN-SCHWARZ SUPERSTRING

E. NISSIMOV*, S. PACHEVA* and S. SOLOMON**

Department of Physics, Weizmann Institute of Science, Rehovot 76100, Israel

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We construct a gauge-invariant superspace action in terms of unconstrained off-shell superfields for the $D = 10$ supersymmetric Yang-Mills (SYM) theory. We use to this effect:

(i) the point particle limit of the BRST charge of the covariantly quantized harmonic Green-Schwarz superstring,

(ii) a general covariant action principle for overdetermined systems of nonlinear field equations of motion.

One obtains gauge and super-Poincaré invariant equations of motion equivalent to the Nilsson's constraints for $D = 10$ SYM.

In the previous approaches (light-cone-gauge, component-fields) one would have to sacrifice either explicit Lorentz invariance or explicit supersymmetry while in the present approach they are both manifest.

Unfortunately the action we find is nonlocal in space-time. To restore locality one may have to introduce additional degrees of freedom.

1. Introduction

It is hoped that a relativistic quantum theory of supersymmetric strings [1, 2] can describe in a consistent way the quantum theory of space-time i.e. quantum gravity [3]. Unfortunately the proof (or disproof) of this conjecture was not completed to this date [4] because it was impossible until recently to express the quantum theory of superstrings in a form which displays explicitly the super-Poincaré invariance.

In a series of papers [5–10] this obstacle was overcome through the introduction of appropriate "spinorial vielbein" variables called "harmonic variables". It becomes now appropriate to address the question of how the SYM and SUGRA theories do appear in this explicitly covariant quantum superstring formalism.

In the present paper we use the BFV-BRST [11] charge of the super-Poincaré covariant first quantized GS superstring with $N = 1$ space-time SUSY computed in

* Permanent address: Institute of Nuclear Research and Nuclear Energy, Boul. Lenin 72, 1784 Sofia, Bulgaria.

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[10] to construct the gauge and super-Poincaré covariant field theory corresponding to its zero-mass sector (i.e. the $D=10$ SYM) in terms of unconstrained (off-shell) superfields. To this end we employ (and review below) the general covariant action principle for arbitrary consistent overdetermined systems of nonlinear field equations proposed in our preceding paper [12].

The supersymmetric $D=10$ SYM field theory was discovered in the component formalism by [13, 14] and has an on-shell supersymmetry due to the celebrated Fierz identity for $D=10$ σ-matrices:

$$(\sigma^{\mu})_{\alpha\beta}(\sigma_{\mu})_{\gamma\delta}+(\sigma^{\mu})_{\beta\gamma}(\sigma_{\mu})_{\alpha\delta}+(\sigma^{\mu})_{\gamma\alpha}(\sigma_{\mu})_{\beta\delta}=0. \tag{1.1}$$

Unfortunately, in the form in which it was discovered, the SYM lagrangian:

$$L=-\tfrac{1}{4}\mathrm{tr}(f_{\mu\nu}f^{\mu\nu})-\tfrac{1}{2}i\,\mathrm{tr}(w\,\nabla\!\!\!\!/\,w) \tag{1.2}$$

was not explicitly supersymmetric. In (1.2) w_{α} is a left-handed Majorana-Weyl $D=10$ spinor while $f_{\mu\nu}$ and ∇_{μ} are gauge-covariant expressions in terms of a gauge vector field $a_{\mu}(x)$:

$$igf_{\mu\nu}=[\nabla_{\mu},\nabla_{\nu}],\qquad \nabla_{\mu}=\partial_{\mu}+ig[a_{\mu},\cdot\,].$$

We use here lower case characters a, ∇, f, w, ϕ etc... in order to emphasize that the respective expressions are ordinary fields and *not* superfields as we will use in the rest of the paper and denote by capitals: A, ∇, F, W, ϕ. Later we will also introduce ghost-haunted superfields which we will denote by $\mathscr{A}$, $\mathscr{F}$, $\mathscr{W}$, Φ etc.

The field equations of motion which this field action generates by varying with respect to a and w respectively are: generalizations of the Maxwell and Dirac equations respectively:

$$\nabla^{\mu}f_{\mu\nu}=\tfrac{1}{2}g(\sigma_{\nu})^{\alpha\beta}\{w_{\alpha},w_{\beta}\},\qquad \nabla\!\!\!\!/\,w\equiv(\sigma^{\mu})^{\alpha\beta}\,\nabla_{\mu}w_{\beta}=0. \tag{1.3}$$

In order to obtain an explicitly supersymmetric theory it was tried to formulate the theory in terms of superfields: $\phi(x,\theta)=\phi(x)+\sum_{r=1}^{16}(1/r!)\theta_{\alpha_1}\dots\theta_{\alpha_r}\phi^{\alpha_1\cdots\alpha_r}(x)$. The general 1-form gauge superfield in the $D=10$ $N=1$ superspace:

$$A(x,\theta)=\mathrm{d}x^{\mu}A_{\mu}(x,\theta)+\mathrm{d}\theta_{\alpha}A^{\alpha}(x,\theta) \tag{1.4}$$

describes too many degrees of freedom and in order to describe just the on-shell SYM it has to be submitted to certain constraints (the Nilsson constraints) [15]:

$$gF^{\alpha\beta}\equiv\{\nabla^{\alpha},\nabla^{\beta}\}-2i(\sigma^{\mu})^{\alpha\beta}\,\nabla_{\mu}=0, \tag{1.5}$$

where

$$\nabla_\mu = \partial_\mu + ig\left[A_\mu, \cdot\right], \qquad \nabla^\alpha = D^\alpha + g\left[A^\alpha, \cdot\right\}, \qquad D^\alpha = \frac{\partial}{\partial\theta_\alpha} + i(\sigma^\mu)^{\alpha\beta}\theta_\partial\partial_\mu . \tag{1.6}$$

The integrability conditions (1.5) together with their consequences from the Bianchi identities for the above covariant derivatives can be shown to be equivalent to the ordinary field equations of motion obtained from the (1.2) lagrangian [16–18]. The great puzzle was the fact that it was still impossible to find an explicitly super-Poincaré invariant action from which the Nilsson constraints (1.9) would appear as field equations of motion when the action is varied with respect to the superfields. The exact conditions for this impossibility were codified in certain no-go theorems [19].

The harmonic "spinor-vielbein-like" variables avoid the above no-go theorems [5–10]. This is not completely unexpected in view of similar successes obtained by the harmonic superspace approach in different contexts [20]. Moreover the apparent relation of certain "vielbein-like" auxiliary variables with supertwistors [21–23] renders natural their usefulness in describing massless systems [16, 18].

In the present paper we show that our gauge and super-Poincaré invariant unconstrained superfield action based on the point-particle limit of the BRST charge Q_{BRST} of the super-Poincaré covariant GS superstring gives on-shell the Nilsson constraint equations of $D = 10$ SYM.

The plan of the paper is as follows. In sect. 2 we review pedagogically the developments [5–10] which lead us to the super-Poincaré covariant Q_{BRST} of the GS superstring. In particular we explain the origin of the auxiliary variables and of the additional gauge invariances. Also the statement in the recent paper by Kallosh and Rahmanov [24] claiming "nonunitarity" of our formalism is shown to be incorrect. In sect. 3 we describe the super-Poincaré covariant first quantization of the $N = 1$ BS superparticle [25–29] (the zero-mode of the GS superstring) in the Dirac canonical formalism. Sect. 4 is devoted to the covariant first- and second-quantization of the $D = 10$ $N = 1$ BS superparticle in the BFV-BRST formalism. In sect. 5 we derive a harmonic superfield representation of the Nilsson constraints for $D = 10$ $N = 1$ SYM and prove its equivalence to the original Nilsson constraints. In particular, the linearized form of these harmonic superfield equations is shown to exactly coincide with the Dirac constraint equations for the superfield wave function of the covariantly quantized $D = 10$ $N = 1$ BS superparticle. In sect. 6 we review our general covariant action principle for arbitrary overdetermined systems of nonlinear field equations and apply it to construct a superspace action for $D = 10$ $N = 1$ SYM in terms of unconstrained (off-shell) superfields. In sect. 7 we discuss the implications of the present results and the directions for further developments.

The appendix supplies the general proof of the pure gauge nature of the auxiliary harmonic variables, needed to perform super-Poincaré covariant quantization.

2. The super-Poincaré covariant quantization of the GS superstring

The present work constructs the unconstrained superfield action of the $D=10$ SYM making use crucially of the point-particle limit of the explicitly super-Poincaré invariant BRST charge of the GS superstring. Such a BRST construction was possible as a consequence of the harmonic superstring program for a manifestly super-Poincaré covariant quantization of the GS superstring which we developed during the last year [5–10]. In order to make the structure and the origin of the BRST charge construction clear, we will describe in this section in a sketchy but hopefully pedagogical way the main ideas and concepts of the harmonic superstring program. The BS action in the hamiltonian formalism is:

$$S=\int \mathrm{d}\tau\left[p_\mu\,\partial_\tau x^\mu+p_\theta^\alpha\,\partial_\tau\theta_\alpha-H_T\right], \tag{2.1}$$

$$H_T=\lambda p^2+\lambda_\alpha D^\alpha. \tag{2.2}$$

In (2.1), θ_α is a left-handed $D=10$ MW spinor λ and λ_α are Lagrange multiplies, and the fermionic constraint D^α reads:

$$D^\alpha\equiv -ip_\theta^\alpha-\not{p}^{\alpha\beta}\theta_\beta. \tag{2.3}$$

These constraints are half first-class and half second-class. This was for years the puzzle of covariant quantization of the GS superstring (and the BS superparticle): the spinor objects relevant for the quantization procedure are too small to fit into a spinor representation of the Lorentz group. In fact the structure of the constraints requires objects which transform under the group $\mathrm{SO}(8)\times\mathrm{SO}(1,1)$. We want to be able to express objects transforming as spinors of SO(8) without breaking the SO(1,9) Lorentz symmetry of the system. To this end we introduce the auxiliary variables [5–10] u_μ^a, $v_\alpha^{\pm\frac{1}{2}}$ where the indices μ,α transform as vector and MW spinor under the global Lorentz SO(1,9) respectively, while the indices $a,\pm\frac{1}{2}$ transform respectively under the internal groups SO(8) and SO(1,1). These auxiliary variables will act as "spinorial vielbeins" bridging covariantly between SO(1,9) and SO(8) spinor objects. They fulfill by definition the orthogonality relations:

$$u_\mu^a u^{b\mu}=C^{ab},\qquad \left[v_\alpha^{\pm\frac{1}{2}}(\sigma^\mu)^{\alpha\beta}v_\beta^{\pm\frac{1}{2}}\right]u_\mu^a=0,$$

$$\left[v_\alpha^{+\frac{1}{2}}(\sigma^\mu)^{\alpha\beta}v_\beta^{+\frac{1}{2}}\right]\left[v_\gamma^{-\frac{1}{2}}(\sigma_\mu)^{\gamma\delta}v_\delta^{-\frac{1}{2}}\right]=-1, \tag{2.4}$$

where C^{ab} denotes the invariant metric tensor in the relevant SO(8) representation space.

In the sequel the following two light-like vectors $u_\mu^\pm$ will appear, which are composite variables built out of the elementary variables $v_\alpha^{\pm\frac{1}{2}}$:

$$u_\mu^\pm = v_\alpha^{\pm\frac{1}{2}}(\sigma_\mu)^{\alpha\beta}v_\beta^{\pm\frac{1}{2}}. \tag{2.5}$$

This construction automatically encodes the light-like geometrical character of $u_\mu^\pm$ which is due to the $D=10$ Fierz identity (1.1). There are indications that this fact has deep relations with the twistor light-like geometry of space-time [30, 22, 23]. Henceforth, we shall use the shorthand notations:

$$A^\pm \equiv u_\mu^\pm A^\mu, \qquad A^a \equiv u_\mu^a A^\mu, \qquad \sigma^{a_1\dots a_n} \equiv u_{\mu_1}^{a_1}\dots u_{\mu_n}^{a_n}\sigma^{[\mu_1}\dots\sigma^{\mu_n]} \tag{2.6}$$

for any Lorentz vector A^μ. Let us particularly stress that $A^\pm$, A^a are Lorentz scalars and they should not be confused with the vector components of A^μ which appear in the non-covariant light-cone formalism.

The gauge invariances, insuring that the introduction of the vielbein-like variables u, v does not affect the physics, are expressed in the hamiltonian formalism by the first class constraints:

$$D^{ab} \equiv u_\mu^a \frac{\partial}{\partial u_{\mu b}} - u_\mu^b \frac{\partial}{\partial u_{\mu a}} + \tfrac{1}{2}\left(v^{+\frac{1}{2}}\sigma^{ab}\frac{\partial}{\partial v^{+\frac{1}{2}}} + v^{-\frac{1}{2}}\sigma^{ab}\frac{\partial}{\partial v^{-\frac{1}{2}}}\right), \tag{2.7}$$

$$D^{-+} \equiv \tfrac{1}{2}\left(v_\alpha^{+\frac{1}{2}}\frac{\partial}{\partial v_\alpha^{+\frac{1}{2}}} - v_\alpha^{-\frac{1}{2}}\frac{\partial}{\partial v_\alpha^{-\frac{1}{2}}}\right), \tag{2.8}$$

$$D^{\pm a} \equiv u_\mu^\pm \frac{\partial}{\partial u_{\mu a}} + \tfrac{1}{2}v^{\mp\frac{1}{2}}\sigma^\pm\sigma^a\frac{\partial}{\partial v^{\mp\frac{1}{2}}}. \tag{2.9}$$

They express the fact (analogous to the principle of equivalence) that the physics is invariant under arbitrary rotations of the "vielbein-like" frame $(u_\mu^a, u_\mu^\pm)$. The operators (2.7)–(2.9) represent indeed the SO(1,9) algebra under commutation.

The second important requirement is the requirement about the *specific* independence of the wave functions $\phi(x, \theta, u, v)$ on the auxiliary variables $u_\mu^a, v_\alpha^{\pm\frac{1}{2}}$. The representation space $\mathscr{H}_0$ in which the above quantum operators will act is spanned by definition by functions of the following general form (here ϕ is taken in the momentum space representation with respect to x):

$$\phi(p,\theta,u,v) = \sum_{\{\lambda\}\{\nu\}}\left(\frac{u_{\lambda_1}^+}{p^+}\right)\dots\left(\frac{u_{\lambda_k}^+}{p^+}\right)\left(\frac{u_{\nu_1}^-}{p^-}\right)\dots\left(\frac{u_{\nu_l}^-}{p^-}\right)\phi_{kl}^{\{\lambda\}\{\nu\}}(p,\theta,u,v) \tag{2.10}$$

(recall $p^{\pm} \equiv v^{\pm\frac{1}{2}} \not{p} v^{\pm\frac{1}{2}}$), where $\phi_{kl}^{\{\lambda\}\{\nu\}}$ are defined by the following *specific* expansion in the auxiliary variables u, v:

$$\phi_{kl}^{\{\lambda\}\{\nu\}}(p, \theta, U, v) = \sum_{\{\mu\}\{\kappa\}} \left[u_{\mu_1}^{a_1} \dots u_{\mu_n}^{a_n}\right]_{\mathrm{SO}(8)\ \mathrm{singlet}} u_{\kappa_1}^{+} \dots u_{\kappa_m}^{+} u_{\kappa_{m+1}}^{-} \dots u_{\kappa_{2m}}^{-}$$

$$\times \phi_{klnm}^{\{\lambda\}\{\nu\}\{\mu\}\{\kappa\}}(p, \theta). \tag{2.11}$$

The expansion (2.11) is characterized by the fact that each term is a monomial in the auxiliary variables in which all the SO(8) × SO(1,1) indices are *saturated* among the u_μ^a's and the $v_\alpha^{\pm 1/2}$'s only, whereas the coefficients $\phi_{klmn}^{\{\lambda\}\{\nu\}\{\mu\}\{\kappa\}}(p, \theta)$ are arbitrary ordinary superfields *inert* under the SO(8) × SO(1,1) internal group, i.e. they do not carry any SO(8) × SO(1,1) indices. In order to have the terms of the expansion (2.10) all independent, the coefficient superfields have to be symmetric in the indices $\{\lambda\}$ and $\{\nu\}$:

$$\phi_{kl}^{\{\dots\lambda_i\dots\lambda_j\dots\}\{\nu\}}(p, \theta, u, v) = \phi_{kl}^{\{\dots\lambda_j\dots\lambda_i\dots\}\{\nu\}}(p, \theta, u, v),$$

$$\phi_{kl}^{\{\lambda\}\{\dots\nu_i\dots\nu_j\dots\}}(p, \theta, u, v) = \phi_{kl}^{\{\lambda\}\{\dots\nu_j\dots\nu_i\dots\}}(p, \theta, u, v), \tag{2.12}$$

traceless:

$$\eta_{\lambda_i\lambda_j}\phi_{kl}^{\{\dots\lambda_i\dots\lambda_j\dots\}\{\nu\}}(p, \theta, u, v) = 0, \qquad \eta_{\nu_i\nu_j}\phi_{kl}^{\{\lambda\}\{\dots\nu_i\dots\nu_j\dots\}}(p, \theta, u, v) = 0,$$

$$\eta_{\mu\nu} = \mathrm{diag}(-, +, \dots, +) \tag{2.13}$$

and transverse with respect to p_μ, $u_\mu^{\pm}$:

$$p_{\lambda_i}\phi_{kl}^{\{\dots\lambda_i\dots\}\{\nu\}}(p, \theta, u, v) = 0, \qquad p_{\nu_i}\phi_{kl}^{\{\lambda\}\{\dots\nu_i\dots\}}(p, \theta, u, v) = 0, \tag{2.14}$$

$$\left(\eta_{\mu_i\lambda_i} + u_{\mu_i}^{+} u_{\lambda_i}^{-}\right)\phi_{kl}^{\{\dots\lambda_i\dots\}\{\nu\}}(p, \theta, u, v) = 0,$$

$$\left(\eta_{\mu_i\nu_i} + u_{\mu_i}^{-} u_{\nu_i}^{+}\right)\phi_{kl}^{\{\lambda\}\{\dots\nu_i\dots\}}(p, \theta, U, v) = 0. \tag{2.15}$$

Similarly, in order to have the terms in the expansion (2.11) all independent, the coefficient superfields, besides the properties coming from (2.12)–(2.15), have to obey additionally the properties of symmetry and tracelessness with respect to the indices $\{\kappa\}$:

$$\phi_{klnm}^{\{\lambda\}\{\nu\}\{\mu\}\{\dots\kappa_i\dots\kappa_j\dots\}}(p, \theta) = \phi_{klnm}^{\{\lambda\}\{\nu\}\{\mu\}\{\dots\kappa_j\dots\kappa_i\dots\}}(p, \theta), \tag{2.16}$$

$$\eta_{\kappa_i\kappa_j}\phi_{klnm}^{\{\lambda\}\{\nu\}\{\mu\}\{\dots\kappa_i\dots\kappa_j\dots\}}(p, \theta) = 0.$$

The Dirac constraint equations:

$$D^{ab}\phi = 0\,, \qquad D^{-+}\phi = 0 \tag{2.17}$$

are *identically* satisfied on the space $\mathscr{H}_0$ of wave functions given by (2.10) and (2.11). Therefore, $\phi(p, \theta, u, v)$ (2.10) is actually a function on the homogenous space $\mathscr{L}/\mathrm{SO}(8) \times \mathrm{SO}(1,1)$ instead of being a function on the original space $\mathscr{L}$ defined by the kinematical constraints (2.4) on $v_\alpha^{\pm 1/2}$, u_μ^a. Hence the functions of the form (2.10) will be called *harmonic superfields* whereas the functions (2.11) will be called *analytic harmonic superfields* (because of their analytic dependence on $v_\alpha^{\pm\frac{1}{2}}$, u_μ^a). This also justifies the name "harmonic" for the auxiliary variables $v_\alpha^{\pm\frac{1}{2}}$, u_μ^a which effectively enter the present formalism through $\mathscr{L}/\mathrm{SO}(8) \times \mathrm{SO}(1,1)$. Analytic harmonic superfields were first introduced in a different context in [20].

Let us point out, that the harmonic superfields (2.10), (2.11) are also characterized by the fact that they do not carry external overall $\mathrm{SO}(8) \times \mathrm{SO}(1,1)$ indices (hence the subscript $_0$ in the notation $\mathscr{H}_0$ of their space). In sect. 3 we shall introduce more general harmonic superfields bearing external $\mathrm{SO}(8) \times \mathrm{SO}(1,1)$ indices which are simply expressed in terms of the fields (2.10), (2.11) (see also the appendix).

Now, one can easily deduce (cf. [6]), that the Dirac constraint equations:

$$(D^{ab}, D^{-+}, D^{+a}, D^{-a})\phi = 0 \tag{2.18}$$

on the space $\mathscr{H}_0$ (2.10), (2.11) imply (in the notations of (2.10), (2.11)):

$$\phi_{0000}(p, \theta) = \text{arbitrary}\,,$$

$$\phi_{klnm}^{\{\lambda\}\{\nu\}\{\mu\}\{\kappa\}}(p, \theta) = 0\,, \qquad (k, l, m, n) \neq (0,0,0,0)\,, \tag{2.19}$$

i.e. the Dirac first-class constraints (2.7)–(2.9) *together* with the specification (2.10), (2.11) of the representation space $\mathscr{H}_0$ imply that u_μ^a, $v_\alpha^{\pm\frac{1}{2}}$ are pure gauge degrees of freedom. A simple explanation of this property is that the harmonic superfields from $\mathscr{H}_0$ (2.10), (2.11) depend only on 45 independent combinations of harmonic variables*: $u_\mu^a, u_\mu^\pm = v^{\pm\frac{1}{2}}\sigma_\mu v^{\pm\frac{1}{2}}$ which exactly matches the number of equations (2.18).

The restriction of the quantum states of the harmonic formalism $\phi(p, \theta, u, v)$ to the form (2.10), (2.11), i.e. to the space $\mathscr{H}_0$ is crucial. It is this restriction which substitutes within the harmonic formalism for the "missing"

$$14 \text{ gauge invariances} = 59\left(\text{the number of independent } u_\mu^a, v_\alpha^{\pm\frac{1}{2}} \text{ from (2.4)}\right)$$

$$-45\;(\text{the number of } D^{ab}, D^{-+}, D^{\pm a})\,,$$

* The 100 harmonic combinations u_μ^a, $u_\mu^\pm$ are subject to 55 kinematical constraints (cf. [31]): $u_\mu^a u^{b\mu} = C^{ab}$, $u_\mu^a u^{\pm\mu} = 0$, $(u^\pm)^2 = 0$, $u_\mu^+ u^{-\mu} = -1$.

which *would be* necessary to gauge away completely all the u_μ^a's and the $v_\alpha^{\pm\frac{1}{2}}$'s *if* the wave functions ϕ were allowed to depend *arbitrarily* on the u_μ^a's and the $v_\alpha^{\pm\frac{1}{2}}$'s:

$$\phi_{\text{arbitrary}}(p,\theta,u,v) = \sum_{\{\mu\}\{\alpha\}\{\beta\}} u_{\mu_1}^{a_1}\dots u_{\mu_n}^{a_n} v_{\alpha_1}^{+\frac{1}{2}}\dots v_{\alpha_m}^{\pm\frac{1}{2}} v_{\beta_1}^{-\frac{1}{2}}\dots v_{\beta_k}^{-\frac{1}{2}}$$

$$\times \phi_{a_1\dots a_n}^{+\frac{1}{2}(m-k)\{\mu\}\{\alpha\}\{\beta\}}(p,\theta). \tag{2.20}$$

Overlooking the crucial difference between the naive wave functions (2.20) and the relevant space $\mathscr{H}_0$ of quantum states defined by (2.10), (2.11) leads to a statement in a recent paper by Kallosh and Rahmanov [24] claiming the "non-unitarity" of our formalism. The above explanation, and the discussion in the appendix below, shows that their claim is not correct.

From the constraint algebra one easily deduces the action describing the pure-gauge "dynamics" of the system of harmonic variables u_μ^a, $v_\alpha^{\pm\frac{1}{2}}$:

$$S_{\text{harmonic}} = \int \mathrm{d}\tau \Big[p_{ua}^\mu\, \partial_\tau u_\mu^a + p_v^{-\frac{1}{2}\alpha}\, \partial_\tau v_\alpha^{+\frac{1}{2}} + p_v^{+\frac{1}{2}\alpha}\, \partial_\tau v_\alpha^{-\frac{1}{2}}$$

$$-\Lambda_{ab} D^{ab} - \Lambda^{+-} D^{-+} - \Lambda_a^- D^{+a} - \Lambda_a^+ D^{-a}\Big]. \tag{2.21}$$

In (2.21) $\Lambda_{ab},\dots,\Lambda_a^+$ denote Lagrange multipliers for the corresponding first-class constraints $D^{ab},\dots,D^{-a}$ which are the classical counterparts of the harmonic differential operators (2.7)–(2.9) and, therefore all constraints are first class.

The classical analog of the requirements (2.10), (2.11) on the representation space $\mathscr{H}_0$ of quantum states is the requirement on the form of the classical "observables" [6]. The latter are not arbitrary functions of (u, v) and their conjugate momenta (p_u, p_v), but are given as expansions in $(u, v;\, p_u, p_v)$ where all internal $SO(8)\times SO(1,1)$ are saturated among u, v, p_u, p_v and, therefore the corresponding coefficients do not carry any $SO(8)\times SO(1,1)$ indices.

Let us now see how the presence of the auxiliary variables allows us to express separately the first and second class constraints [6].

Given the constraints (2.3), and using the vielbein-like harmonic variables (2.4) we can express the 16-component 10 D MW spinor constraint (2.3):

$$D^\alpha \equiv -ip_\theta^\alpha - \not{p}^{\alpha\beta}\theta_\beta$$

in terms of 2 sets of 8 Lorentz scalars (organized as two SO(8) spinors):

$$G^{+\frac{1}{2}a} = \tfrac{1}{2} u_\mu^a v_\alpha^{-\frac{1}{2}} (\sigma^\mu)^{\alpha\beta} (\sigma^+)_{\beta\gamma} D^\gamma, \tag{2.22}$$

$$D^{+\frac{1}{2}a} = u_\mu^a v_\alpha^{+\frac{1}{2}} (\sigma^\mu)^{\alpha\beta} \not{p}_{\beta\gamma} D^\gamma. \tag{2.23}$$

The projectors in front of D^{γ} are chosen such that the Lorentz indices are saturated. In particular one sees that without the v's it is impossible to saturate the Lorentz-spinor indices. However $v_{\alpha}^{\pm\frac{1}{2}}$ together with $(\sigma^{\mu})^{\alpha\beta}$ can convert a spinor index $_{\beta}$ into a vector index $^{\mu}$ which is then saturated with an u_{μ}^{a}. The role of σ^{+} (cf. (2.6), $+$ is an internal SO(1,1) index which is inert under Lorentz transformations) in (2.22) is to appropriately raise a spinor index. The role of $\not{p}$ in (2.23) is to make $D^{+\frac{1}{2}a}$ first class.

It turns out that indeed $G^{+\frac{1}{2}a}$ constitute 8 second class constraints (i.e. the matrix of their Poisson brackets

$$\{G^{+\frac{1}{2}a}, G^{+\frac{1}{2}b}\}_{\mathrm{PB}} = ip^{+}C^{ab} \tag{2.24}$$

is nonsingular) while (2.23) are first class (i.e.:

$$\{D^{+\frac{1}{2}a}, D^{+\frac{1}{2}b}\}_{\mathrm{PB}} = -2i(p^{+})p^{2} \qquad (=0 \text{ modulo constraints})). \tag{2.25}$$

In turn D^{α} can be reconstructed out of $D_{a}^{+\frac{1}{2}}$ and $G_{a}^{+\frac{1}{2}}$:

$$D^{\alpha} = (p^{+})^{-1}(\sigma^{b}v^{+\frac{1}{2}})^{\alpha}D_{b}^{+\frac{1}{2}} + (p^{+})^{-1}(\not{p}\sigma^{+}\sigma^{b}v^{-\frac{1}{2}})^{\alpha}G_{b}^{+\frac{1}{2}} \tag{2.26}$$

Once the covariant separation of the constraints is effectuated, one can use a trick invented in [32, 33] to transform the second class constraints $G^{+\frac{1}{2}a}$ (2.22) into first class constraints $\hat{G}^{+\frac{1}{2}a}$ without changing the physical content of the constrained system* by introducing auxiliary dynamical *real* fermionic variables Ψ^{a} [8, 9] with transformation properties** and Poisson brackets similar to the second class constraints which they convert into first class ones:

$$\hat{G}^{+\frac{1}{2}a} = G^{+\frac{1}{2}a} + \sqrt{p^{+}}\,\Psi^{a}, \tag{2.27}$$

$$\{\Psi^{a}, \Psi^{b}\}_{\mathrm{PB}} = -iC^{ab}. \tag{2.28}$$

One can then use the decomposition (2.26) to reconstruct [8, 9] the Lorentz MW-spinor first-class constraint $\hat{D}^{\alpha}$ out of the first class constraints $D^{+\frac{1}{2}a}$ (2.23) and $\hat{G}^{+\frac{1}{2}a}$ (2.27):

$$\begin{aligned}\hat{D}^{\alpha} &\equiv (p^{+})^{-1}(\sigma^{b}v^{+\frac{1}{2}})^{\alpha}D_{b}^{+\frac{1}{2}} + (p^{+})^{-1}(\not{p}\sigma^{+}\sigma^{b}v^{-\frac{1}{2}})^{\alpha}\hat{G}_{b}^{+\frac{1}{2}} \\ &\equiv D^{\alpha} + (p^{+})^{-\frac{1}{2}}(\not{p}\sigma^{+}\sigma^{a}v^{-\frac{1}{2}})^{\alpha}\Psi_{a}.\end{aligned} \tag{2.29}$$

The introduction of the auxiliary fermionic variables Ψ^{a} (2.28) into (2.29) necessitates the simultaneous modification of the harmonic first class constraints

* In some special cases [23] the Faddeev-Shatashvili trick works even when separation is not possible. However, the auxiliary vielbein-like variables u, v are always necessary. It turns out that they are related with certain twistor-like $D = 10$ objects [23].

** The Ψ's are related to the Grassmann-odd components of certai supertwistors [23].

(2.7)–(2.9):

$$D^{ab} \to \hat{D}^{ab} = D^{ab} + \tilde{R}^{ab}, \tag{2.30}$$

$$D^{-a} \to \hat{D}^{-a} = D^{-b} - \frac{p_b}{p^+}\tilde{R}^{ab}, \tag{2.31}$$

with D^{-+}, D^{+a} remaining the same, where:

$$\tilde{R}^{ab} \equiv \tfrac{1}{2}(\tilde{S}^{ab})_{cd}\Psi^c\Psi^d, \tag{2.32}$$

$$(\tilde{S}^{ab})_{cd} \equiv \tfrac{1}{2}(\tilde{\gamma}^{ab})_{cd} \equiv \tfrac{1}{2}v^{-\frac{1}{2}}\sigma_c\sigma^{ab}\sigma^+\sigma_d v^{-\frac{1}{2}}. \tag{2.33}$$

The 8×8 matrices $\tilde{S}^{ab}$ (2.33) are precisely the $D = 10$ Lorentz-invariant generators of the harmonic SO(8) (c)-spinor representation (see [10]).

Also one can easily check, using the explicit expression (2.32) and the anticommutation (upon quantization) relations (2.28), that:

$$[\tilde{R}^{ab}, \tilde{R}^{cd}] = C^{bc}\tilde{R}^{ad} - C^{ac}\tilde{R}^{bd} + C^{ad}\tilde{R}^{bc} - C^{bd}\tilde{R}^{ac}, \tag{2.34}$$

$$[D^{ab}, \tilde{R}^{cd}] = 0. \tag{2.35}$$

Thus, both parts D^{ab} (2.7) and $\tilde{R}^{ab}$ (2.32) in the modified first-class constraint $\hat{D}^{ab}$ (2.30) (generating once again the SO(8) algebra under commutation) may be interpreted as harmonic "orbital" and harmonic "spin" SO(8) rotations respectively. The implications of these will be elaborated upon in sect. 3 (see also the appendix).

The modifications (2.30), (2.31) are needed in order to preserve the first-class property of the new system of covariant and irreducible constraints:

$$p^2, \quad \hat{D}^{\alpha}(2.29), \quad \hat{D}^{ab}(2.30), \quad D^{-+}, \quad D^{+a}, \quad \hat{D}^{-a}(2.31), \tag{2.36}$$

Thus we arrive at the harmonic BS action [9]:

$$\hat{S}_{\text{superparticle}} = \hat{S}_{\text{BS}} + \hat{S}_{\text{harmonic}}, \tag{2.37}$$

$$\hat{S}_{\text{BS}} = \int \mathrm{d}\tau \left[p_\mu\,\partial_\tau x^\mu + p_\theta^\alpha\,\partial_\tau\theta_\alpha + i\Psi^a\partial_\tau\Psi_a - \Lambda p^2 - \Lambda_\alpha\hat{D}^\alpha \right], \tag{2.38}$$

$$\hat{S}_{\text{harmonic}} = \int \mathrm{d}\tau \Big[p_{u_a}^\mu\,\partial_\tau u_\mu^a + p_v^{-\frac{1}{2}\alpha}\,\partial_\tau v_\alpha^{+\frac{1}{2}} + p_v^{+\frac{1}{2}\alpha}\,\partial_\tau v_\alpha^{-\frac{1}{2}}$$

$$- \Lambda_{ab}\hat{D}^{ab} - \Lambda^{+-}D^{-+} - \Lambda_a^- D^{+a} - \Lambda_a^+\hat{D}^{-a} \Big]. \tag{2.39}$$

The new action (2.37)–(2.39) is physically equivalent to the original BS action (2.1), however, it possesses the decisive advantage of having super-Poincaré covariant and irreducible first-class constraints only. Thus, the super-Poincaré covariant canonical quantization of the BS superparticle (either à la Dirac or in the BFV-BRST formalism) is now straightforward (see sects. 3 and 4).

Let us particularly stress, that all first-class constraints (2.36) and the auxiliary fermionic variables Ψ_a (2.28) are all *real*. Therefore, the harmonic BS action (2.37)–(2.39) is *real* too. The harmonic BS and GS actions in [8–10, 23] are also *real*.

The harmonic super-string action in the hamiltonian formulation is a generalization of the above harmonic BS action (2.37)–(2.39) [8–10]:

$$\hat{S} = \hat{S}_{\mathrm{GS}} + \hat{S}_{\mathrm{harmonic}}, \tag{2.40}$$

$$\hat{S}_{\mathrm{GS}} = \int \mathrm{d}\tau \int_{-\pi}^{\pi} \mathrm{d}\xi \Big[P_\mu \partial_\tau X^\mu + \sum_{A=1,2} \left(p_{\theta A}^{\alpha} \partial_\tau \theta_{A\alpha} + i \Psi_A^a \partial_\tau \Psi_{Aa} \right) - \sum_{A=1,2} \left(\Lambda_A \hat{T}_A + \Lambda_{A\alpha} \hat{D}_A^\alpha \right) \Big]. \tag{2.41}$$

The main characteristics of this harmonic superstring action (2.40) (2.41) are:

(i) it contains the harmonic space variables $v_\alpha^{\pm\frac{1}{2}}$, u_μ^a;

(ii) it contains new fermionic string variables $\Psi_A^a(\xi)$;

(iii) all its constraints are first class and irreducible;

(iv) the space-time supersymmetry is realized linearly;

(v) it possesses a larger set of gauge invariances and it reduces, in a particular gauge, to the original GS action.

The term $\hat{S}_{\mathrm{harmonic}}$ in (2.40) has precisely the same form as S_{harmonic} (2.21) with the constraints D^{ab}, D^{-a} appropriately modified due to the introduction of $\Psi_A^a(\xi)$ (cf. (2.30) (2.31)). Accordingly, the new first class, independent and covariant system of constraints is more complicated [10]. The constraints generalizing the harmonic constraints (2.7)–(2.9) are:

$$D^{-+} \equiv \tfrac{1}{2}\left(v_\alpha^{+\frac{1}{2}} \frac{\partial}{\partial v_\alpha^{+\frac{1}{2}}} - v_\alpha^{-\frac{1}{2}} \frac{\partial}{\partial v_\alpha^{-\frac{1}{2}}} \right), \tag{2.42}$$

$$D^{+a} \equiv u_\mu^+ \frac{\partial}{\partial u_{\mu a}} + \tfrac{1}{2} v^{-\frac{1}{2}} \sigma^+ \sigma^a \frac{\partial}{\partial v^{-\frac{1}{2}}}, \tag{2.43}$$

$$\hat{D}^{ab} \equiv D^{ab} + \sum_A \int_{-\pi}^{\pi} \mathrm{d}\xi\, \tilde{R}_A^{ab}, \tag{2.44}$$

$$\hat{D}^{-a} \equiv D^{-a} - \sum_A \int_{-\pi}^{\pi} \mathrm{d}\xi \left(\Pi_A^+\right)^{-1} \Pi_{Ab} \tilde{R}_A^{ab} - \tfrac{1}{3} i \sum_A (-1)^A \int_{-\pi}^{\pi} \mathrm{d}\xi \left(\Pi_A^+\right)^{-\frac{3}{2}} \tilde{R}_A^{cd} \left(v^{-\frac{1}{2}} \sigma_b \sigma^a \sigma_{cd} \sigma^+ \theta_A' \right) \Psi_A^b, \tag{2.45}$$

where (cf. (2.32)):

$$\tilde{R}_A^{ab} \equiv \tfrac{1}{2} \left(\tilde{S}^{ab}\right)_{cd} \Psi_A^c \Psi_A^d, \tag{2.46}$$

$$\left\{ \Psi_A^a(\xi), \Psi_B^b(\eta) \right\}_{\mathrm{PB}} = -i \delta_{AB} C^{ab} \delta(\xi - \eta) \qquad (\text{cf. } (2.28)), \tag{2.47}$$

and $\tilde{S}^{ab}$ is the same as in (2.33). The bosonic constraints generalizing p^2 are:

$$\hat{T}_A(\xi) \equiv \Pi_A^2 - 4i(-1)^A \theta'_{A\alpha} D_A^\alpha + 2i(-1)^A \Psi_A^a(\xi) \Psi'_{A\alpha}(\xi), \tag{2.48}$$

with the notation:

$$\Pi_A^\mu \equiv P^\mu + (-1)^A [X'^\mu + 2i\theta_A \sigma^\mu \theta'_A]. \tag{2.49}$$

The fermionic constraints (2.29) are generalized by:

$$\hat{D}_A^\alpha(\xi) \equiv D_A^\alpha(\xi) - i(-1)^A (\Pi_A^+)^{-1} (\sigma^{bc}\sigma^+ \theta'_A)^\alpha \tilde{R}_{Abc}$$
$$+ (\Pi_A^+)^{-\frac{1}{2}} (\not{\Pi}_A \sigma^+ \sigma^b v^{-\frac{1}{2}})^\alpha \Psi_{Ab}, \tag{2.50}$$

where D_A^α is the mixture of first and second class constraints appearing in the original GS formulation [1]:

$$D_A^\alpha = -ip_{\theta A}^\alpha - [P^\mu + (-1)^A (X'^\mu + i\theta_A \sigma^\mu \theta'_A)](\sigma_\mu \theta_A)^\alpha.$$

The information of the new set of covariant, BFV-irreducible, first class constraints is encoded in the BRST charge [10]:

$$Q_{\mathrm{BRST}} = \hat{Q}_{\mathrm{harmonic}} + Q_{\mathrm{string}}; \tag{2.51}$$

$$\begin{aligned}
\hat{Q}_{\mathrm{harmonic}} = i\eta_{ab} \Bigg[& \hat{D}^{ab} + \eta^{+a} \frac{\partial}{\partial \eta_b^+} - \eta^{+b} \frac{\partial}{\partial \eta_a^+} \\
& + \eta^{-a} \frac{\partial}{\partial \eta_b^-} - \eta^{-b} \frac{\partial}{\partial \eta_a^-} + \eta_d^a \frac{\partial}{\partial \eta_{bd}} - \eta_d^b \frac{\partial}{\partial \eta_{ad}} \Bigg] \\
& + i\eta^{+-} \left[D^{-+} + \eta_a^+ \frac{\partial}{\partial \eta_a^+} - \eta_a^- \frac{\partial}{\partial \eta_a^-} \right] + i\eta_a^- D^{+a} \\
& + i\eta_a^+ \Bigg[\hat{D}^{-a} + \eta_a^- \frac{\partial}{\partial \eta^{+-}} - \eta^{-b} \frac{\partial}{\partial \eta^{ab}} \\
& + \tfrac{1}{2} \sum_A \int_{-\pi}^{\pi} \mathrm{d}\xi (\Pi_A^+)^{-2} \left(\eta^{+b} \tilde{R}^{Aab} - i(\Pi_A^+)^{\frac{1}{2}} (\chi_A \sigma^+ \sigma_{ab} v^{-\frac{1}{2}}) \Psi_A^b \right) \\
& \times \left(\frac{\delta}{\delta c_A} + 4i(-1)^A \theta'_{A\alpha} \frac{\delta}{\delta \chi_{A\alpha}} \right) \Bigg],
\end{aligned} \tag{2.52}$$

$$\begin{aligned}
Q_{\mathrm{string}} = \sum_A \int_{-\pi}^{\pi} \mathrm{d}\xi \Bigg\{ & c_A \left[\hat{T}_A - 4i(-1)^A \left(c'_A \frac{\delta}{\delta c_A} + \chi'_{A\alpha} \frac{\delta}{\delta \chi_{A\alpha}} \right) \right] + \chi_{A\alpha} \hat{D}_A^\alpha \\
& + (2\Pi_A^+)^{-1} (\chi_A \sigma^+ \chi_A) \left[\frac{\delta}{\delta c_A} + 4i(-1)^A \theta'_{A\alpha} \frac{\delta}{\delta \chi_{A\alpha}} \right] \Bigg\}.
\end{aligned} \tag{2.53}$$

The ghosts $c_A, \chi_{A\alpha}, \eta_{ab}, \eta^{+-}, \eta^{-a}, \eta^{+a}$, appearing in (2.52)–(2.53) correspond respectively to the constraints $\hat{T}_A$, $\hat{D}^\alpha_A$, $\hat{D}^{ab}$, D^{-+}, D^{+a}, $\hat{D}^{-a}$.

We conclude this section with the following remarks concerning the extension [24] of the harmonic superstring program [5–10] to the lagrangian formalism. This important development might allow for the application of the powerful methods of the two-dimensional conformal field theory to the covariantly quantized GS superstring.

Since in the present formalism the harmonic variables $u^a_\mu, v^{\pm\frac{1}{2}}_\alpha$ do not depend on the string world-sheet coordinate ξ, the action $\hat{S}_{\text{harmonic}}$ in (2.40) does not possess manifest reparametrization invariance. However, as already explained in refs. [5–7], the harmonics $v^{\pm\frac{1}{2}}_\alpha$, u^a_μ, whose dynamics is described by the action S_{harmonic} (2.21), are pure-gauge degrees of freedom and, therefore, their independence on the world-sheet parameter ξ does not spoil the reparametrization invariance of the physical superstring dynamics described by (2.41). In fact, in the hamiltonian framework (in which we always work) the reparametrization invariance is accounted for by the presence of the first-class constraints $\hat{T}_A(\xi)$ (2.48), satisfying the correct Virasoro algebra. Therefore, there is no breaking of reparametrization invariance in the present canonical hamiltonian formalism. Moreover, as stressed in [10, 23] nothing prevents us from taking the harmonic auxiliary variables v, u to depend also on ξ by a straightforward generalization of (2.21), (2.42)–(2.45). In the latter case, however, the expressions for the modified superstring constraints $\hat{T}_A$ (2.48), $\hat{D}^\alpha_A$ (2.50) and the PB algebra become more complicated.

Actually, if we delete the fermionic string variables Ψ^a_A from the harmonic GS action (2.40), (2.41) and work instead (as in our original ref. [5]) with the covariantly disentangled first class ($D^{+\frac{1}{2}a}_A \equiv v^{+\frac{1}{2}}\sigma^a/\Pi_A D_A$) and second class ($G^{+\frac{1}{2}a}_A \equiv \frac{1}{2}v^{-\frac{1}{2}}\sigma^a\sigma^+ D_A$) constraints, then it is possible to rewrite S_{harmonic} (2.21) in a manifestly reparametrization invariant form by promoting $v^{\pm\frac{1}{2}}_\alpha$, u^a_μ to depend also on ξ [24].

The set of auxiliary variables used in ref. [24] exactly corresponds to the harmonic variables (2.4) introduced in [5–10] while the constraints in [24] are identical to a subset of the harmonic constraints in [7]*.

Actually, using the auxiliary variables $v_{A\alpha}$, $\tilde{v}^\alpha_A$ introduced in [27], one can construct a simpler manifestly reparametrization invariant harmonic GS action. Here the symbols $A \equiv (a, \dot{a})$, $B \equiv (b, \dot{b})$, $a, b, \dot{a}, \dot{b} = 1, \dots, 8$ label pairs of Lorentz-invariant internal SO(8) (s) and (c)-spinor indices. The explicit form of $S_{\text{auxiliary}}$

* The harmonic variables used by Kallosh and Rahmanov in [24] $v^{\pm 1/2}_\alpha$, u^k_μ, $u^{\dot{k}}_\mu$, ($k, \dot{k} = 1, \dots, 4$) correspond to the harmonic variables $v^{\pm 1/2}_\alpha$, u^a_μ, w^k_a, $\overline{w}^{\dot{k}}_a$, ($k, \dot{k} = 1, \dots, 4$) of ref. [7] through the relation $u^k_\mu = w^k_a u^a_\mu$, $u^{\dot{k}}_\mu = \overline{w}^{\dot{k}}_a u^a_\mu$. The sets of harmonic constraints $\{H\}, \{F\}, \{K\}$ in [24] correspond to $\{D^{-+}$ (eq. (2.8) above), $E^{IJ}, E^{+-}\}, \{D^{+a}$(eq. (2.9)), $E^{+I}\}, \{(\frac{1}{2}v^{+1/2}\sigma^{ab}\partial/\partial v^{+1/2})$ part of D^{ab}(eq. (2.7))$\}$ of [7]. Here E^{IJ}, E^{+-}, E^{+I} are the "second generation" harmonic constraints involving $w^k_a, \overline{w}^k_a$ which helped us in [7] to reduce covariantly SO(8) to SU(4) $\times$ U(1).

entering the modified GS action

$$\hat{S}_{\mathrm{GS}} = S_{\mathrm{GS}} + S_{\mathrm{auxiliary}} \tag{2.54}$$

reads:

$$S_{\mathrm{auxiliary}} = \int \mathrm{d}\tau\, \mathrm{d}\xi \sqrt{-g}\left[\left(\tilde{p}^{\alpha}_{A} \right)_{z} \partial_{\bar{z}} v^{A}_{\alpha} + \left(p^{A}_{\alpha} \right)_{z} \partial_{\bar{z}} \tilde{v}^{\alpha}_{A} - \mu^{AB}_{z\bar{z}} \Psi_{AB} - \lambda^{AB}_{\bar{z}} \left(\mathscr{D}_{AB} \right)_{z} \right], \tag{2.55}$$

where Ψ_{AB} and $\mathscr{D}_{AB}$ [27]:

$$\Psi_{AB}(\tau, \xi) \equiv v_{A\alpha}(\tau, \xi)\, \tilde{v}^{\alpha}_{B}(\tau, \xi) - \delta_{AB} = 0, \tag{2.56}$$

$$\left(\mathscr{D}_{AB} \right)_{z} \equiv v_{A\alpha}\left(\tilde{p}^{\alpha}_{B} \right)_{z} - \tilde{v}^{\alpha}_{A}\left(p_{B\alpha} \right)_{z} \tag{2.57}$$

are 2×256 Lorentz-covariant and functionally independent Dirac first-class constraints, responsible for the pure-gauge nature of the 2×256 auxiliary variables $v_{A\alpha}$, $\tilde{v}^{\alpha}_{A}$.

In (2.55) (2.57) z, $\bar{z}$ denote (anti)self-dual world-sheet indices defined through the 2-dimensional world-sheet (anti)self-duality projectors [2]:

$$P^{nm}_{+} \equiv \tfrac{1}{2}\left(g^{mn} + \frac{\epsilon^{nm}}{\sqrt{-g}} \right) = e^{n}_{z} e^{m}_{\bar{z}}, \qquad P^{nm}_{-} \equiv \tfrac{1}{2}\left(g^{mn} - \frac{\epsilon^{nm}}{\sqrt{-g}} \right) = e^{n}_{\bar{z}} e^{m}_{z},$$

where $e^{n}_{z,\bar{z}}$ are world-sheet zweibeins corresponding to the world-sheet metric g_{mn}. Then:

$$\left(p^{\alpha}_{A} \right)_{z} = e^{n}_{z}\left(p^{\alpha}_{A} \right)_{n}, \qquad e^{n}_{\bar{z}}\left(p^{\alpha}_{A} \right)_{n} = 0, \qquad \lambda^{AB}_{\bar{z}} = e^{n}_{\bar{z}} \lambda^{AB}_{n}, \qquad e^{n}_{z} \lambda^{AB}_{n} = 0,$$

and similarly for $\mu_{z,\bar{z}}$.

With the help of the auxiliary dynamical variables $(v^{A}_{\alpha}, \tilde{v}^{\alpha}_{A})$, we can now express the fermionic κ-gauge invariance [26] of $\hat{S}$ (2.54) in a Lorentz-covariant and irreducible way:

$$\delta_{\kappa} \theta_{\alpha} = i\left(\Pi\!\!\!/_{\bar{z}} \right)_{\alpha\beta} \tilde{v}^{\beta}_{a} \kappa^{a}_{z} \tag{2.58}$$

where the gauge parameter κ^{a}_{z} has only 8 (and therefore – *independent*) Lorentz invariant components*.

One can continue covariantly the quantization procedure in the lagrangian formalism by imposing covariant gauge-fixing condition for the irreducible κ-gauge

* Recall $a = 1, \ldots, 8$ which is the correct number of independent κ-gauge symmetries (cf. [28]).

symmetry (2.58) (cf. [34]):

$$\chi_a \equiv \tilde{v}_a^{\alpha}\theta_\alpha = 0\,. \tag{2.59}$$

The corresponding gauge fixing in the hamiltonian formalism was used in [10] in the process of constructing the covariant vertices.

By further imposing the gauge conditions [24]:

$$\lambda_{\bar{z}}^{AB} = \mu_{z\bar{z}}^{AB} = 0$$

one may obtain a gauge fixed action of the form:

$$\hat{S}_{\mathrm{GS}}|_{\text{gauge fixed}} = S_{\mathrm{GS}}|_{\tilde{v}\theta=0} + S_{\text{auxiliary}}|_{\mu=\lambda=0} + (\text{ghost terms})\,. \tag{2.60}$$

Let us emphasize that (2.60) is *not* manifestly super-Poincaré invariant since the gauge fixing condition (2.59) apparently breaks half of the space-time SUSY. Consequently, the supersymmetry algebra becomes nonlinearly realized as in the non-covariant light-cone formalism [2].

Constructing systems as (2.54)–(2.57) or [24] in which the number of new constraints equals the number of new auxiliary variables is not difficult [27, 23] and it is esthetically appealing★ but it is not a necessary condition for the consistency of the model. This was already shown in [6] for our case and it is well known in general from the harmonic superspace approach [20, 31]. Namely, the "missing" gauge symmetries are substituted in the harmonic superspace approach by the requirement for specific dependence of the superfields on the auxiliary variables (2.4). For additional details, see the appendix.

The lagrangian formulations of the type (2.54) are useful if one wants to quantize covariantly the GS superstring within the lagrangian functional-integral approach [35]. However, for our main objective: an explicitly space-time supersymmetric superstring quantum field theory, it is preferable to use (as we do in the present work) the hamiltonian formalism and a set of variables which are *strongly* confined on the harmonic constraint shell (2.4).

3. Super-Poincaré covariant quantization à la Dirac of the BS superparticle

Before entering in the details of the construction of our gauge covariant and manifestly super-Poincaré covariant field theory for $D = 10$ SYM, we discuss the first quantized theory of the zero-mode (point-particle) limit of the GS superstring, i.e. the ($N = 1$) BS superparticle.

★ Using such systems, in which the auxiliary variables are not strongly constrained by equations of the type (2.4), one obtained interesting relations between supersymmetric particles, twistors and higher N SYM in 4 dimensions [23].

In the present section we describe the super-Poincaré covariant first quantization of the latter in the Dirac canonical formalism.

The physical states of the $D=10$ $N=1$ BS superparticle are the ones which fulfill the Dirac constraint equations (as it will be explained below, they are matrix equations for a vector-valued ϕ in our representation space)

$$p^2\phi = 0\,, \tag{3.1}$$

$$\hat{D}^{\alpha}\phi \equiv \left[D^{\alpha} + \frac{1}{\sqrt{p^+}}\left(\not{p}\sigma^+\sigma^b v^{-\frac{1}{2}}\right)\Psi_b\right]\phi = 0\,, \tag{3.2}$$

$$\hat{D}^{ab}\phi \equiv \left[u^a_\mu \frac{\partial}{\partial u_{\mu b}} - u^b_\mu \frac{\partial}{\partial u_{\mu a}} + \tfrac{1}{2}\left(v^{+\frac{1}{2}}\sigma^{ab}\frac{\partial}{\partial v^{+\frac{1}{2}}} + v^{-\frac{1}{2}}\sigma^{ab}\frac{\partial}{\partial v^{-\frac{1}{2}}}\right) + \tilde{R}^{ab}\right]\phi = 0\,, \tag{3.3}$$

$$D^{-+}\phi \equiv \tfrac{1}{2}\left(v^{+\frac{1}{2}}_\alpha \frac{\partial}{\partial v^{+\frac{1}{2}}_\alpha} - v^{-\frac{1}{2}}_\alpha \frac{\partial}{\partial v^{-\frac{1}{2}}_\alpha}\right)\phi = 0\,, \tag{3.4}$$

$$D^{+a}\phi \equiv \left(u^+_\mu \frac{\partial}{\partial u_{\mu a}} + \tfrac{1}{2}v^{-\frac{1}{2}}\sigma^+\sigma^a\frac{\partial}{\partial v^{-\frac{1}{2}}}\right)\phi = 0\,, \tag{3.5}$$

$$\hat{D}^{-a}\phi \equiv \left[\left(u^-_\mu \frac{\partial}{\partial u_{\mu a}} + \tfrac{1}{2}v^{+\frac{1}{2}}\sigma^-\sigma^a\frac{\partial}{\partial v^{+\frac{1}{2}}}\right) - (p^+)^{-1}p_b\tilde{R}^{ab}\right]\phi = 0\,, \tag{3.6}$$

where

$$\tilde{R}^{ab} \equiv \tfrac{1}{4}\left(\tilde{\gamma}^{ab}\right)_{cd}\Psi^c\Psi^d \tag{3.7}$$

and the linear operators of the left-hand side of (3.1)–(3.6) are the quantized first class constraints (2.36).

In passing to second quantization one reinterprets the quantum states as classical fields, and the constraint equation (3.1)–(3.6) as the free field equations (recall that the superparticle hamiltonian is weakly zero).

We will use a matrix representation with respect to Ψ^a and a functional representation with respect to the other variables $z=(x^\mu, \theta_\alpha, u^a_\mu, v^{\pm\frac{1}{2}}_\alpha)$. In the following, we will call each vector of functions representing a quantum state a "wave function" for conciseness. Let us explicitate the matrix structure of (3.1)–(3.6) following from the matrix representation of the quantum operators corresponding to the variables Ψ^a.

The Grassmann variables Ψ^a are defined in (2.27) to fulfill Poisson brackets relations (2.28) which at the quantum level determine the anticommutation relations of the corresponding operators (which we denote also by Ψ^a). According to these anticommutation relations, the operators Ψ^a form an 8-dimensional Clifford algebra:

$$\{\Psi^a, \Psi^b\} = C^{ab}. \tag{3.8}$$

Therefore the operators Ψ^a can be faithfully and irreducibly represented by 16×16 SO(8) Dirac Γ-matrices:

$$\Psi^a = \sqrt{\tfrac{1}{2}}\,\Gamma_8^a. \tag{3.9}$$

The index a of Ψ^a transforms under the SO(8) generators (3.3) according to the relation:

$$[\hat{D}^{ab}, \Psi^c] = -(\tilde{S}^{ab})_{cd}\Psi^d, \tag{3.10}$$

$$(\tilde{S}^{ab})_{cd} \equiv \tfrac{1}{2}(\tilde{\gamma}^{ab})_{cd} = v^{-\frac{1}{2}}\sigma_c\sigma^{ab}\sigma^+\sigma_d v^{-\frac{1}{2}}. \tag{3.11}$$

Consequently, the Ψ^a are in the harmonic (c)-spinor representation. Due to the triality properties of the harmonic SO(8) representations, the Ψ^a's will relate states which are in the harmonic (s) representation to states in the harmonic (v) representation. Moreover, since Ψ^a are grassman-odd, they will relate bosons to fermions. In conclusion, the Ψ^a's are represented by the 16×16 matrices:

$$\Psi^a = \sqrt{\tfrac{1}{2}}\,\Gamma_8^a = \begin{bmatrix} 0 & \sqrt{\tfrac{1}{2}}\,(\gamma^a)_{bc} \\ \sqrt{\tfrac{1}{2}}\,(\tilde{\gamma}^a)_{bc} & 0 \end{bmatrix}, \tag{3.12}$$

where

$$(\gamma^a)_{bc} \equiv \sqrt{2}\,v^{+\frac{1}{2}}\sigma_b\sigma^a\sigma_c v^{-\frac{1}{2}}, \qquad (\tilde{\gamma}^a)_{bc} \equiv \sqrt{2}\,v^{-\frac{1}{2}}\sigma_b\sigma^a\sigma_c v^{+\frac{1}{2}} \tag{3.13}$$

are the Lorentz-invariant harmonic $D=8$ σ-matrices [10]. Ψ^a act on states of the form (recall $z \equiv (x^\mu, \theta_\alpha, u_\mu^a, v_\alpha^{\pm\frac{1}{2}})$):

$$\phi(z) = \begin{bmatrix} F^a(z) \\ B^a(z) \end{bmatrix}, \tag{3.14}$$

where F^a are fermions and B^a are bosons. Let us stress that the wave function $\phi(z)$ (3.14) is *real* (only in this case it will describe on shell the $D=10$ SYM multiplet; see sect. 5).

The internal SO(8) rotation properties of these objects are obtained by looking at how they are acted upon by the "non-orbital" $\tilde{R}^{ab}$ part in $\hat{D}^{ab}$ (recall (2.34), (2.35)):

$$\tilde{R}^{ab} = \tfrac{1}{4}(\tilde{\gamma}^{ab})_{cd}\Psi^{[c}\Psi^{d]} = \begin{bmatrix} \left(\tfrac{1}{4}(\tilde{\gamma}^{ab})_{cd}\tfrac{1}{2}(\gamma^{cd}\right) & 0 \\ 0 & \left(\tfrac{1}{4}(\tilde{\gamma}^{ab})_{cd}\tfrac{1}{2}(\tilde{\gamma}^{cd}\right) \end{bmatrix}$$

$$= \begin{bmatrix} S'^{ab} & 0 \\ 0 & V^{ab} \end{bmatrix}, \tag{3.15}$$

where

$$(V^{ab})_{cd} = C^{ac}C^{bd} - C^{ad}C^{bd} \tag{3.16}$$

is the harmonic SO(8) (v) representation [10] and S'^{ab} is a representation related to the harmonic SO(8) (s) representation S^{ab} (see [10]):

$$(S^{ab})_{cd} = \tfrac{1}{2}(\gamma^{ab})_{cd} = \tfrac{1}{2}v^{\pm\frac{1}{2}}\sigma_c\sigma^{ab}\sigma^-\sigma_d v^{+\frac{1}{2}} \tag{3.17}$$

through a similarity transformation U:

$$S'^{ab} = US^{ab}U^{-1}, \tag{3.18}$$

$$[U]^{ab} \equiv \sqrt{2}\,r_c(\gamma^c)^{ab} \equiv 2(v^{+\frac{1}{2}}\sigma^a\sigma^c\sigma^b v^{-\frac{1}{2}})(v^{+\frac{1}{2}}\sigma_c v^{-\frac{1}{2}}) = C^{ab} - 4r^a r^b, \tag{3.19}$$

$$r_c \equiv v^{+\frac{1}{2}}\sigma_c v^{-\frac{1}{2}}. \tag{3.20}$$

Consequently, we can now write the Dirac constraint equations (3.1)–(3.6) defining the physical quantum states in our matrix representation. The harmonic constraint equations are:

$$(\hat{D}^{ab}\phi)^c \equiv [(D^{ab} + \tilde{R}^{ab})\phi]^c = \begin{bmatrix} D^{ab}F^c + (S'^{ab})^c_{\ d}F^d \\ D^{ab}B^c + (V^{ab})^c_{\ d}B^d \end{bmatrix} = 0, \tag{3.21}$$

$$([D^{-+}\phi]^c = \begin{bmatrix} D^{-+}F^c \\ D^{-+}B^c \end{bmatrix} = 0, \tag{3.22}$$

$$([D^{+a}\phi]^c = \begin{bmatrix} D^{+a}F^c \\ D^{+a}B^c \end{bmatrix} = 0, \tag{3.23}$$

$$(\hat{D}^{-a}\phi)^b \equiv \left[\left(D^{-a} - \tilde{R}^{ac}\frac{p_c}{p^+}\right)\phi\right]^b = \begin{bmatrix} D^{-a}F^b - \dfrac{p_c}{p^+}(S'^{ac})^b_{\ d}F^d \\ D^{-a}B^b - \dfrac{p_c}{p^+}(V^{ac})^b_{\ d}B^d \end{bmatrix} = 0. \tag{3.24}$$

In view of (3.18), in order to have F^a transform in the standard (s) representation and also, in order to absorb the factor $1/\sqrt{p}^{+}$ in (3.2) it is natural to work with new superfield wave functions $\phi'(z)$ which are obtained from $\phi(z)$ through the following linear transformation:

$$\phi(z) \to \phi'(z) = \begin{bmatrix} \sqrt{p}^{+} U^{ab} & 0 \\ 0 & C^{ab} \end{bmatrix} \begin{bmatrix} F_b(z) \\ B_b(z) \end{bmatrix} = \begin{bmatrix} Y^{+\frac{1}{2}a}(z) \\ B^a(z) \end{bmatrix}. \tag{3.25}$$

In terms of $\phi'(z)$ (3.25), the Dirac constraint equations (3.1)–(3.6) acquire the form:

$$(-\partial^2)\phi' \equiv \begin{bmatrix} (-\partial^2) Y^{+\frac{1}{2}a} \\ (-\partial^2) B^a \end{bmatrix} = 0, \tag{3.26}$$

$$\hat{D}^{\alpha}\phi' \equiv \begin{bmatrix} D^{\alpha} Y^{+\frac{1}{2}a} - i\left(\not{\partial}\sigma^b\sigma^a v^{+\frac{1}{2}}\right)^{\alpha} B_b \\ D^{\alpha}B^a - \dfrac{1}{\partial^+}\left(\not{\partial}\sigma^a\sigma^b v^{+\frac{1}{2}}\right)^{\alpha} Y_b^{+\frac{1}{2}} \end{bmatrix}. \tag{3.27}$$

$$D^{+a}\phi' \equiv \begin{bmatrix} D^{+a} Y^{+\frac{1}{2}a} \\ D^{+a} B^a \end{bmatrix} = 0, \tag{3.28}$$

$$\hat{D}^{-a}\phi' \equiv \begin{bmatrix} \left(D^{-a} - \dfrac{1}{2}\dfrac{\partial^a}{\partial^+}\right) Y^{+\frac{1}{2}b} - \dfrac{\partial_c}{\partial^+}(S^{ac})^b_{\ d} Y^{+\frac{1}{2}d} \\ D^{-a}B^b - \dfrac{\partial_c}{\partial^+}(V^{ac})^b_{\ d} B^d \end{bmatrix} = 0, \tag{3.29}$$

$$\hat{D}^{ab}\phi' \equiv \begin{bmatrix} D^{ab} Y^{+\frac{1}{2}c} + (S^{ab})^c_{\ d} Y^{+\frac{1}{2}d} \\ D^{ab}B^c + (V^{ab})^c_{\ d} B^d \end{bmatrix} = 0, \tag{3.30}$$

$$\hat{D}^{-+}\phi' \equiv \begin{bmatrix} \left(D^{-+} - \frac{1}{2}\right) Y^{+\frac{1}{2}a} \\ D^{-+} B^a \end{bmatrix} = 0, \tag{3.31}$$

where D^{ab}, D^{-+}, $D^{\pm a}$ are as in (2.7)–(2.9) and $\partial^+ \equiv u^+_\mu \partial^\mu$, $\partial^a \equiv u^a_\mu \partial^\mu$. Henceforth, the prime on ϕ will be omitted.

The constraint equations (3.30), (3.31) express the fact that the wave function $\phi(z)$ (3.25) is a harmonic $SO(8) \times SO(1,1)$ invariant. This is natural generalization of the properties of the harmonic superfields belonging to the space $\mathscr{H}_0$ defined by (2.10), (2.11).

The harmonic superfields (2.10) identically satisfied the harmonic equations (2.17) where the harmonic "spin" part $\tilde{R}^{ab}$ (3.15) is absent, since (2.10) do not carry external overall SO(8) × SO(1, 1) indices unlike the case of $\phi(z)$ (3.25). Therefore, it is natural to call $Y^{+\frac{1}{2}a}(z)$, $B^a(z)$ *harmonic superfields with external* SO(8) × SO(1, 1) *indices* (see also the appendix). In order to see their structure, one has to actually solve (3.30), (3.31) explicitly.

To achieve this, one expresses first the functions $Y^{+\frac{1}{2}a}(z)$ and $B^a(z)$ in terms of new functions $Y_\alpha(z)$ and $B^\mu(z)$ which carry external Lorentz indices but *not* external SO(8) × SO(1, 1) harmonic indices:

$$Y^{+\frac{1}{2}a} = \left(v^{+\frac{1}{2}}\sigma^a\right)^\alpha Y_\alpha, \tag{3.32}$$

$$B^a = u^a_\mu B^\mu. \tag{3.33}$$

In terms of the new functions, the equations (3.30) and (3.31) reduce to the requirement that the fields Y_α, B^μ are invariant under the orbital SO(8) × SO(1, 1) rotations of $v^{\pm\frac{1}{2}}_\alpha$, u^a_μ

$$D^{-+}\begin{bmatrix} Y_\alpha \\ B^\mu \end{bmatrix} = 0, \tag{3.34}$$

$$D^{ab}\begin{bmatrix} Y_\alpha \\ B_\mu \end{bmatrix} = 0, \tag{3.35}$$

i.e. $Y_\alpha(z)$, $B^\mu(z)$ are general harmonic superfields (without external SO(8) × SO(1, 1) indices) belonging to the space $\mathscr{H}_0$ specified by (2.10), (2.11). The representation (3.32), (3.33) is unique because the harmonic objects u^a_μ and $(v^{+\frac{1}{2}}\sigma^a)^\alpha$ have exactly the same "internal" SO(8) × SO(1, 1) properties as $B^a(z)$ and $Y^{+\frac{1}{2}a}(z)$ (recall eqs. (3.30), (3.31)):

$$D^{ab}\left(v^{+\frac{1}{2}}\sigma^c\right)^\alpha = -\left(S^{ab}\right)^c_d\left(v^{+\frac{1}{2}}\sigma^d\right)^\alpha, \qquad D^{ab}u^\alpha_\mu = -\left(V^{ab}\right)^c_d u^d_\mu \tag{3.36}$$

and, moreover, the objects u^a_μ and $(v^{+\frac{1}{2}}\sigma^a)^\alpha$ are the only harmonic objects to have the property (3.36) (and the correct SO(1, 1) charges).

From now on we shall work only on the space $\mathscr{H}$ of superfield wave functions of the form given by (3.32), (3.33)

$$\phi(z) = \begin{bmatrix} \left(v^{+\frac{1}{2}}\sigma^a\right)^\alpha Y_\alpha(z) \\ u^a_\mu B^\mu(z) \end{bmatrix}. \tag{3.37}$$

Since on $\mathscr{H}$ (3.37) the Dirac constraint equations (3.30), (3.31) are fulfilled identically, i.e. realized operatorially, $\hat{D}^{ab}$ and $\hat{D}^{-+}$ can be dropped from among the set of Dirac constraints to be imposed on the physical states. The remaining constraints $(-\partial^2)$, $\hat{D}^{\alpha}$, D^{+a}, $\hat{D}^{-a}$ will be imposed only "weakly" as conditions on the physical states (3.26)–(3.29). In order to analyze their implications it is useful to perform the following transformation on $Y_\alpha(z)$ and $B^\mu(z)$ in (3.37):

$$\begin{bmatrix} Y_\alpha \\ B^\mu \end{bmatrix} \to \begin{bmatrix} A^\alpha \\ A^\mu \end{bmatrix},$$

$$Y_\alpha(z) = \tfrac{1}{2} i\, \partial^+ (\sigma^-)_{\alpha\beta} \left[A^\beta(z) + i D^\beta \lambda(z) \right], \tag{3.38}$$

$$B^\mu(z) = A^\mu(z) + \partial^\mu \lambda(z), \tag{3.39}$$

where

$$\lambda(z) \equiv -\int^{x^-} u^+_\mu A^\mu(x(y^-; u, v), \theta, u, v)\, \mathrm{d}y^-, \tag{3.40}$$

$$x^- \equiv u^-_\mu x^\mu, \qquad x^\mu(y^-; u, v) \equiv (\eta^{\mu\nu} + u^{+\mu} u^{-\nu}) x_\nu - u^{+\mu} y^-, \qquad \partial^+ \equiv u^+_\mu \partial^\mu .$$

Inserting (3.38)–(3.40) into (3.37) one can easily show that the Dirac constraint equations (3.26)–(3.29) for the covariantly quantized $N = 1$ BS superparticle result in the linearized Nilsson constraint equations of the free $D = 10$ $N = 1$ SYM for A^α, A^μ which become independent on (u, v).

This statement, instead of being directly proved here, will arise as a simple consequence of the more general considerations in the sect. 5.

4. Covariant BFV-BRST first- and second-quantization of the BS superparticle

In this section we perform the super-Poincaré covariant first-quantization of the $N = 1$ $D = 10$ BS superparticle in the BFV-BRST formalism and indicate its equivalence with the canonical Dirac quantization of the preceding question. We also write down a superspace free-field action for the linearized $D = 10$ SYM in terms of unconstrained superfields yielding as equations of motion the Dirac constraint equations (3.26)–(3.29) for the superfield wave function $\phi(z)$ of the $N = 1$ BS superparticle.

From the mathematical point of view the Dirac system (3.26)–(3.29) is an overdetermined system of 33 matrix equations (33 = number $\mathcal{N}$ of Dirac constraints $(-\partial^2)$, $\hat{D}^{\alpha}$, D^{+a}, $\hat{D}^{-a}$) for only one vector-valued function $\phi(z)$. This overdetermined system is however consistent (integrable) since the linear operators $(-\partial^2)$, $\hat{D}^{\alpha}$, D^{+a}, $\hat{D}^{-a}$ acting on $\phi(z)$ (3.37) form a closed algebra under (anti-)commutation:

$$\{\hat{D}^{\alpha}, \hat{D}^{\beta}\} = -i\frac{(\sigma^{+})^{\alpha\beta}}{\partial^{+}}(-\partial^{2}), \tag{4.1}$$

$$[\hat{D}^{-a}, \hat{D}^{-b}] = \frac{1}{(\partial^{+})^{2}}\begin{bmatrix} S^{ab} & 0 \\ 0 & V^{ab}\end{bmatrix}(-\partial^{2}), \tag{4.2}$$

$$[\hat{D}^{-a}, \hat{D}^{\alpha}] = \begin{bmatrix} 0 & U\gamma_{b} \\ \frac{i}{\partial^{+}}\tilde{\gamma}_{b}U & 0\end{bmatrix}\frac{i}{2\sqrt{2}}(\sigma^{+}\sigma^{ab}v^{-\frac{1}{2}})^{\alpha}\frac{1}{\partial^{+}}(-\partial^{2}), \tag{4.3}$$

$$[D^{+a}, \hat{D}^{-b}] = C^{ab}\hat{D}^{-+} + \hat{D}^{ab} = 0 \qquad (\text{on the space } \mathscr{H}\ (3.37)), \tag{4.4}$$

the rest of the commutators being identically zero. Here, once again the notations (3.13), (3.16), (3.17), (3.19) were used.

The BRST charge Q_0 corresponding to the operator algebra (4.1)–(4.4) precisely coincides with the zero mode (point particle) limit of Q_{BRST} (2.51)–(2.53) of the harmonic GS superstring, where the contributions of $\hat{D}^{ab}$ and D^{-+} are deleted (because we are working on the space $\mathscr{H}$ (3.37) of harmonic superfields). We write Q_0 in matrix form:

$$Q_0 = \begin{bmatrix} Q_0^{(YY)}C^{ab} & [Q_0^{(YB)}]^{ab} \\ [Q_0^{(BY)}]^{ab} & Q_0^{(BB)}C^{ab}\end{bmatrix}, \tag{4.5}$$

$$\begin{aligned} Q_0^{(YY)} = {} & c(-\partial^{2}) + \chi_{\alpha}D^{\alpha} - (2i\partial^{+})^{-1}(\chi\sigma^{+}\chi)\frac{\partial}{\partial c} + i\eta_{a}^{-}D^{+a} \\ & + i\eta_{a}^{+}\left[D^{-a} - \frac{1}{2}\frac{\partial^{a}}{\partial^{+}} - \frac{\partial_{b}}{\partial^{+}}S^{ab} - \tfrac{1}{2}(\partial^{+})^{-2}\eta_{b}^{+}S^{ab}\frac{\partial}{\partial c}\right], \end{aligned} \tag{4.6}$$

$$\begin{aligned} Q_0^{(BB)} = {} & c(-\partial^{2}) + \chi_{\alpha}D^{\alpha} - (2i\partial^{+})^{-1}(\chi\sigma^{+}\chi)\frac{\partial}{\partial c} + i\eta_{a}^{-}D^{+a} \\ & + i\eta_{a}^{+}\left[D^{-a} - \frac{\partial_{b}}{\partial^{+}}V^{ab} - \tfrac{1}{2}(\partial^{+})^{-2}\eta_{b}^{+}V^{ab}\frac{\partial}{\partial c}\right], \end{aligned} \tag{4.7}$$

$$[Q_0^{(YB)}]^{ab} = -i\left(\chi\,\partial\!\!\!/\sigma^{b}\sigma^{a}v^{+\frac{1}{2}}\right) + \frac{i\eta_{c}^{+}}{\sqrt{2}\,(2\partial^{+})}(\chi\sigma^{+}\sigma^{cd}v^{-\frac{1}{2}})(U\gamma_{d})^{ab}\frac{\partial}{\partial c}\Big\}, \tag{4.8}$$

$$[Q_0^{(BY)}]^{ab} = -\frac{1}{\partial^{+}}\left(\chi\,\partial\!\!\!/\sigma^{a}\sigma^{b}v^{+\frac{1}{2}}\right) - \frac{\eta_{c}^{+}\sqrt{2}}{(2\partial^{+})^{2}}(\chi\sigma^{+}\sigma^{cd}v^{-\frac{1}{2}})(\tilde{\gamma}_{d}U)^{ab}\frac{\partial}{\partial c}\Big\}. \tag{4.9}$$

The whole information about the algebra (4.1)–(4.4) is simply encoded in the nilpotency property of Q_0 (4.5)–(4.9):

$$Q_0^2 = 0 . \tag{4.10}$$

In the BFV-BRST formalism Q_0 (4.5)–(4.9) is a linear operator acting on the space $\tilde{\mathscr{H}}$ of ghost-haunted harmonic superfields. $\tilde{\mathscr{H}}$ consists of fields of the following form (cf. (3.37)):

$$\Phi(z,\eta) \equiv \begin{bmatrix} \mathscr{Y}^{+\frac{1}{2}a}(z,\eta) \\ \mathscr{B}^{a}(z,\eta) \end{bmatrix} \tag{4.11}$$

with the short-hand notations:

$$z \equiv \left(x^{\mu}, \theta_{\alpha}, u^{a}_{\mu}, v^{\pm\frac{1}{2}}_{\alpha}\right), \qquad \eta \equiv \left(\eta^{A}\right) \equiv \left(c, \chi_{\alpha}, \eta^{\pm a}\right). \tag{4.12}$$

The property that Φ (4.11) is a SO(8) × SO(1,1) harmonic invariant is now expressed by the requirement that the ghost-haunted generators of SO(8) × SO(1,1) annihilate $\Phi(z,\eta)$ (these equations replace (3.30), (3.31) which were fulfilled in the space $\mathscr{H}$ (3.37)):

$$\left(\hat{D}^{-+} + \eta^{+a}\frac{\partial}{\partial\eta^{+a}} - \eta^{-a}\frac{\partial}{\partial\eta^{-a}}\right)\begin{bmatrix}\mathscr{Y}\\ \mathscr{B}\end{bmatrix} = 0, \tag{4.13}$$

$$\left(\hat{D}^{ab} + \eta^{+a}\frac{\partial}{\partial\eta^{+}_{b}} - \eta^{+b}\frac{\partial}{\partial\eta^{+}_{a}} + \eta^{-a}\frac{\partial}{\partial\eta^{-}_{b}} - \eta^{-b}\frac{\partial}{\partial\eta^{-}_{a}}\right)\begin{bmatrix}\mathscr{Y}\\ \mathscr{B}\end{bmatrix} = 0. \tag{4.14}$$

The explicit form of $\Phi(z,\eta)$ satisfying (4.13), (4.14) is given (in complete analogy with (3.32), (3.33)) as:

$$\mathscr{Y}^{+\frac{1}{2}a}(z,\eta) = \left(v^{+\frac{1}{2}}\sigma^{a}\right)^{\alpha}\mathscr{Y}_{\alpha}(z,\eta), \qquad \mathscr{B}^{a}(z,\eta) = u^{a}_{\mu}\mathscr{B}^{\mu}(z,\eta), \tag{4.15–16}$$

$$\mathscr{Y}_{\alpha}(z,\eta) = \sum_{\{\lambda\}\{\nu\}}\left(\frac{u^{+}_{\lambda_1}}{p^{+}}\right)\cdots\left(\frac{u^{+}_{\lambda_n}}{p^{+}}\right)\left(\frac{u^{-}_{\nu_1}}{p^{-}}\right)\cdots\left(\frac{u^{-}_{\nu_n}}{p^{-}}\right)\mathscr{Y}^{\{\lambda\}\{\nu\}}_{(n,m)\alpha}(z,\eta), \tag{4.17}$$

$$\mathscr{B}^{\mu}(z,\eta) = \sum_{\{\lambda\}\{\nu\}}\left(\frac{u^{+}_{\lambda_1}}{p^{+}}\right)\cdots\left(\frac{u^{+}_{\lambda_n}}{p^{+}}\right)\left(\frac{u^{-}_{\nu_1}}{p^{-}}\right)\cdots\left(\frac{u^{-}_{\nu_n}}{p^{-}}\right)\mathscr{B}^{\mu\{\lambda\}\{\nu\}}_{(n,m)}(z,\eta). \tag{4.18}$$

Each coefficient field in the expansions (4.17), (4.18) is an arbitrary analytic harmonic ghost-haunted superfield whose formal expansion in terms of $v^{\pm\frac{1}{2}}_{\alpha}$, u^{a}_{μ}

now reads (cf. (2.11)):

$$\mathscr{X}(z,\eta) = \sum_{\{\mu\}\{\nu\}\{\lambda\}} \left[u^{a_1}_{\mu_1} \dots u^{a_n}_{\mu_n} \eta^{+b_1} \dots \eta^{+b_m} \eta^{-c_1} \dots \eta^{-c_l} \right]_{\mathrm{SO(8)\ singlet}}$$

$$\times u^+_{\nu_1} \dots u^+_{\nu_k} u^-_{\lambda_1} \dots u^-_{\lambda_{k+m-l}} \mathscr{X}^{\{\mu\}\{\nu\}\{\lambda\}}(x,\theta,c,\chi_\alpha), \tag{4.19}$$

where $\mathscr{X}$ stands for any $\mathscr{B}^{\mu\{\lambda\}\{\nu\}}_{(n,m)}$ or $\mathscr{Y}^{\{\lambda\}\{\nu\}}_{(n,m)\alpha}$ which appear in the right-hand-side of the expansions (4.18), (4.17).

One can now perform a transformation of $\mathscr{Y}^{+\frac{1}{2}a}$, $\mathscr{B}^a$ in complete analogy with (3.38)–(3.40) and rewrite (4.11), (4.15), (4.16) in the form:

$$\Phi(z,\eta) \equiv \begin{bmatrix} \mathscr{Y}^{+\frac{1}{2}a}(z,\eta) \\ \mathscr{B}^a(z,\eta) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i}{2}\partial^+(v^{+\frac{1}{2}}\sigma^a\sigma^-)_\beta[\mathscr{A}^\beta(z,\eta) + iD^\beta\lambda(z,\eta)] \\ u^a_\mu[\mathscr{A}^\mu(z,\eta) = \partial^\mu\lambda(z,\eta)] \end{bmatrix}, \tag{4.20}$$

where $\lambda(z,\eta)$ is a functional of $\mathscr{A}^\mu(z,\eta)$ defined in complete analogy with eq. (3.40):

$$\lambda(z) \equiv -\int^{x^-} u^+_\mu \mathscr{A}^\mu(x(y^-;u,v),\theta,u,v;\eta)\,\mathrm{d}y^- . \tag{4.21}$$

The original harmonic superfield $\phi(z)$ (3.37) enters in the ghost-haunted harmonic superfield $\Phi(z,\eta)$ (4.11) as the zeroth order term in the ghost expansion:

$$\Phi(z,\eta) = \phi(z) + \sum_{n\geq 1} \frac{1}{n!}\eta^{A_1}\dots\eta^{A_n}\phi_{A_1\dots A_n}(z). \tag{4.22}$$

Eq. (4.20) together with (4.22) implies:

$$\mathscr{A}^\alpha(z,\eta) = A^\alpha(z) + \sum_{n\geq 1}\frac{1}{n!}\eta^{B_1}\dots\eta^{B_n}A^\alpha_{B_1\dots B_n}(z),$$

$$\mathscr{A}^\mu(z,\eta) = A^\mu(z) + \sum_{n\geq 1}\frac{1}{n!}\eta^{B_1}\dots\eta^{B_n}A^\mu_{B_1\dots B_n}(z), \tag{4.23}$$

where $A^\alpha(z)$, $A^\mu(z)$ are the $D = 10$ SYM supergauge potentials.

In what follows it will be very useful to employ the following condensed notations for the linear generators (3.26)–(3.29) and their respective ghosts:

$$\begin{bmatrix} L_A & | & -\partial^2 & \hat{D}^\alpha & D^{+a} & \hat{D}^{-a} \\ \eta^A & | & c & \chi_\alpha & \eta^{-a} & \eta^{+a} \end{bmatrix}. \tag{4.24}$$

In terms of (4.24), the algebra (4.1)–(4.4) and the BRST charge (4.5)–(4.9) are written short-hand as:

$$[L_A, L_B\} \equiv L_A L_B + (-1)^{\epsilon_A \epsilon_B + 1} L_B L_A = f^C_{AB} L_C , \tag{4.25}$$

$$Q_0 = \eta^A L_A + \tfrac{1}{2}(-1)^{\epsilon_B} \eta^B \eta^C f^A_{CB} \frac{\partial}{\partial \eta^A} . \tag{4.26}$$

In (4.25)–(4.26) ϵ_A denote the Grassmann parity of L_A. The corresponding ghosts η^A have accordingly the opposite parity $\epsilon(\eta^A) = \epsilon_A + 1$.

The key ingredient of the canonical BFV-BRST formalism [11] is that one can rewrite the consistent overdetermined system of (matrix) Dirac constraint equations (3.26)–(3.29) for $\phi(z)$ (3.37) as a single linear matrix equation for $\Phi(z, \eta)$ (4.13):

$$Q_0 \Phi(z, \eta) = 0 . \tag{4.27}$$

An important property of (3.60) is that it possesses a ghost-haunted gauge invariance as a consequence of the nilpotency of Q_0 (4.10)*:

$$\delta_\Lambda \Phi(z, \eta) = Q_0 \Lambda(z, \eta) . \tag{4.28}$$

A fundamental result of the BFV-BRST quantization is the general theorem [11] about the equivalence of the BFV-BRST physical state conditions (4.27), (4.28) with the Dirac constraint equations for the physical wave function (using notations (4.22), (4.25), (4.26)):

$$L_A \phi(z) = 0 , \qquad A = 1, \dots, \mathcal{N} . \tag{4.29}$$

Here is a brief illustration of the above general theorem. Indeed, inserting the ghost expansion of $\Phi(z, \eta)$ (4.22) and the similar expansion for the gauge parameter $\Lambda(z, \eta)$ in (4.28):

$$\Lambda(z, \eta) = \Lambda_0(z) + \sum_{n \geq 1} \frac{1}{n!} \eta^{A_1} \dots \eta^{A_n} \Lambda_{A_1 \dots A_n}(z) \tag{4.30}$$

* Due to nilpotency of Q_0 (4.10) $\Lambda(z, \eta)$ is defined itself only modulo transformation of the type (4.28): $\Lambda \sim \Lambda + Q_0 \Lambda'$ for arbitrary $\Lambda'(z, \eta)$.

into (4.27) and (4.28) and employing the condensed notations (4.24)–(4.26), one obtains:

$$L_A\phi(z) = 0, \qquad \text{(short hand for the Dirac system (3.26)–(3.29))}, \tag{4.31}$$

$$\delta_\Lambda\phi(z) = 0, \tag{4.32}$$

$$L_A\phi_B(z) + (-1)^{\epsilon_A\epsilon_B+1}L_B\phi_A(z) - f^C_{AB}\phi_C(z) = 0, \tag{4.33}$$

$$\delta_\Lambda\phi_A(z) = L_A\Lambda_0(z), \tag{4.34}$$

$$\Big[(-1)^{\epsilon_A\sum_{i=1}^n(\epsilon_{B_i}+1)}\Big(L_A\phi_{B_1\dots B_n}(z) - \tfrac{1}{2}n(-1)^{(\epsilon_C+\epsilon_{B_1})\sum_{j=2}^n(\epsilon_{B_j}+1)} \times f^C_{AB_1}\phi_{CB_2\dots B_n}(z)\Big)\Big]_{\text{antisymm}(A,B_1,\dots,B_n)} = 0, \tag{4.35}$$

$$\delta_\Lambda\phi_{B_1B_2\dots B_n}(z) = \Big[(-1)^{\epsilon_{B_1}\sum_{j=2}^n(\epsilon B_j+1)}\Big(L_{B_1}\Lambda_{B_2\dots B_n}(z) - \tfrac{1}{2}(n-1)(-1)^{(\epsilon_C+\epsilon_{B_2})\sum_{k=3}^n(\epsilon_{B_k}+1)} \times f^C_{B_1B_2}\Lambda_{CB_3\dots B_n}(z)\Big)\Big]_{\text{antisymm}(B_1,\dots,B_n)}, \tag{4.36}$$

for general n. Antisymmetrization in (4.35), (4.36) is defined as:

$$\mathscr{M}_{\dots AB\dots} = (-1)^{(\epsilon_A+1)(\epsilon_B+1)}\mathscr{M}_{\dots BA\dots}\,. \tag{4.37}$$

Now, using (4.25) in the equivalent form:

$$\Big[(-1)^{\epsilon_A(\epsilon_B+1)}\Big(L_AL_B - \tfrac{1}{2}f^C_{AB}L_C\Big)\Big]_{\text{antisymm}(AB)} = 0 \tag{4.38}$$

one can easily check that the general solutions of (4.33), (4.35) are pure-gauge ones (cf. [11]):

$$\begin{aligned}
\phi_A(z) &= \delta_\Lambda\phi_A(z) \qquad \text{(eq. (4.31)) for arbitrary } \Lambda_0(z),\\
\phi_{B_1\dots B_n}(z) &= \delta_\Lambda\phi_{B_1\dots B_n}(z) \qquad \text{(eq. (4.33)) for arbitrary } \Lambda_{B_1\dots B_{n-1}}(z),
\end{aligned} \tag{4.39}$$

whereas the zeroth order term $\phi(z)$ in the ghost expansion (4.22) is gauge-invariant (4.32) and satisfies the canonical system of Dirac constraint equations (4.31).

Now, after establishing the equivalence between the BFV-BRST quantization scheme (eqs. (4.27), (4.28)) and the canonical Dirac formalism (eq. (4.31)), we can write down a field theory action principle yielding the whole overdetermined set of

Dirac constraint equations (4.31) as equations of motion. To this end it is sufficient to construct an action which to generate (4.27) as variation equation and to possess ghost-haunted gauge invariance under the transformation (4.28). The action, we are looking for, reads:

$$S_0 = \tfrac{1}{2}\int \mathrm{d}z\,\mathrm{d}\eta\, \hat{H}\Phi(z,\eta) Q_0 \Phi(z,\eta)\,. \tag{4.40}$$

Here $\hat{H}$ is a linear operator fulfilling the properties (T denotes operator transposition)

$$\hat{H}^{\mathrm{T}} = \hat{H}\,, \qquad Q_0^{\mathrm{T}}\hat{H} = \hat{H}Q_0\,. \tag{4.41}$$

Now, (4.41) together with the nilpotency of Q_0 (4.10) assure the invariance of S_0 (4.40) under the gauge transformation (4.28). Taking into account the explicit expression of Q_0 (4.5)–(4.9) we find the following form of $\hat{H}$ for the case of interest-second quantized $N=1$ BS superparticle or, equivalently, free $D=10$ $N=1$ SYM:

$$\hat{H} = \begin{bmatrix} -\tfrac{1}{2}\left(K_1 + K_1^{\mathrm{T}}\right)\dfrac{1}{\partial^+} & 0 \\ 0 & \tfrac{1}{2}\left(K_2 + K_2^{\mathrm{T}}\right) \end{bmatrix}, \tag{4.42}$$

where $K_{1,2}$ act on the arguments of the corresponding functions $\mathscr{Y}^{+\frac{1}{2}a}(z,\eta)$ and $\mathscr{B}^a(z,\eta)$ from (4.11) as follows:

$$\begin{aligned} K_1:&\quad v_\alpha^{\pm\frac{1}{2}} \to \pm i v_\alpha^{\pm\frac{1}{2}}\,, \qquad c \to -c\,, \qquad \eta^{\pm a} \to -\eta^{\pm a}\,, \\ K_2:&\quad v_\alpha^{\pm\frac{1}{2}} \to \pm i v_\alpha^{\pm\frac{1}{2}}\,, \qquad \chi_\alpha \to -\chi_\alpha\,. \end{aligned} \tag{4.43}$$

Thus, formula (4.40) is the superspace action for the linearized $D=10$ SYM in terms of unconstrained (off-shell) superfields which possesses a Witten's type [36] BFV gauge invariance (4.28).

5. Harmonic superfield representation for the Nilsson SYM constraints

As we have already discussed in sect. 1, the complete on-shell description of $D=10$ $N=1$ SYM theory is given by the Nilsson constraint equations [15–18]:

$$F^{\alpha\beta} \equiv \frac{1}{g}\left(\{\nabla^\alpha, \nabla^\beta\} - 2i\,\nabla\!\!\!\!/^{\,\alpha\beta}\right) = 0\,. \tag{5.1}$$

We will use the standard notations:

$$F^{\alpha\mu} \equiv D^{\alpha}A^{\mu} + i\,\partial^{\mu}A^{\alpha} + g[A^{\alpha}, A^{\mu}],$$

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} + ig[A^{\mu}, A^{\nu}]. \tag{5.2}$$

The fundamental fields in the above equations are $A^{\mu}(x,\theta)$ – the vector superfield gauge potential and $A^{\alpha}(x,\theta)$ – the superfield Majorana-Weyl spinor gauge potential. g denotes the coupling constant.

The Bianchi identities for $\nabla^{\alpha}, \nabla^{\mu}$ are in fact the consistency conditions for the overdetermined nonlinear system (5.1). Multiple application of these identities yields as a consequence of (5.1) the following additional equations for A^{α}, A^{μ} [17,18]:

$$F^{\alpha\mu} - (\sigma^{\mu}W)^{\alpha} = 0, \tag{5.3}$$

$$\nabla^{\alpha}F^{\mu\nu} = ((\sigma^{\mu}\nabla^{\nu} - \sigma^{\nu}\nabla^{\mu})W)^{\alpha}, \tag{5.4}$$

$$\nabla^{\alpha}W_{\beta} = -\tfrac{1}{2}i(\sigma^{\mu\nu})_{\beta}{}^{\alpha}F_{\mu\nu}, \tag{5.5}$$

$$\nabla^{\mu}F_{\mu\nu} = gW\sigma_{\nu}W, \tag{5.6}$$

$$\nabla\!\!\!/ W = 0, \tag{5.7}$$

where W_{α} is a Majorana-Weyl spinor defined by (5.3).

Our aim now is to transform the nonlinear system (5.1), (5.3)–(5.7) into an *equivalent* system of nonlinear equations in terms of harmonic superfields such that the linearized form of the latter to coincide exactly with the system of Dirac constraint equations (3.26)–(3.29) for the wave function of the covariantly quantized $D=10$ $N=1$ BS superparticle. This will provide the complete proof that the covariantly quantized $D=10$ $N=1$ harmonic BS superparticle (2.37)–(2.39) describes on-shell the (linearized) $D=10$ SYM multiplet.

To this end we regard A^{α}, A^{μ} in (5.1), (5.3)–(5.7) as *harmonic superfields* (2.10)–(2.11), i.e. as functions on the extended superspace $z=(x^{\mu}, \theta_{\alpha}, u^{a}_{\mu}, v^{\pm\frac{1}{2}}_{\alpha})$ [5–10] identically satisfying

$$(D^{ab}, D^{-+})\begin{bmatrix} A^{\alpha}(z) \\ A^{\mu}(z) \end{bmatrix} = 0. \tag{5.8}$$

In order to insure the on-shell independence of A^{α}, A^{μ} on the auxiliary harmonic variables (u, v) we add the harmonic differential equations:

$$D^{\pm a}\begin{bmatrix} A^{\alpha}(z) \\ A^{\mu}(z) \end{bmatrix} = 0 \tag{5.9}$$

(cf. the discussion in sect. 2 leading to eqs. (2.18), (2.19); the harmonic differential operators D^{ab}, D^{-+}, $D^{\pm a}$ appearing in (5.8), (5.9) are the same as in (2.7)–(2.9)).

Now, let us consider the following nonlinear field transformation:

$$\begin{bmatrix} A^{\alpha}(z) \\ A^{\mu}(z) \end{bmatrix} \to \phi(z) = \begin{bmatrix} Y^{+\frac{1}{2}a}(z) \\ B^{a}(z) \end{bmatrix}, \tag{5.10}$$

$$Y^{+\frac{1}{2}a}(z) = \tfrac{1}{2}i\left(v^{+\frac{1}{2}}\sigma^{a}\sigma^{-}\right)_{\alpha}\partial^{+}\left[\Omega^{-1}(z)A^{\alpha}(z)\Omega(z) + \frac{1}{g}\Omega^{-1}(z)D^{\alpha}\Omega(z)\right], \tag{5.11}$$

$$B^{a}(z) = u^{a}_{\mu}\left[\Omega^{-1}(z)A^{\mu}(z)\Omega(z) - \frac{i}{g}\Omega^{-1}(z)\,\partial^{\mu}\Omega(z)\right] \tag{5.12}$$

(here $\partial^{+} \equiv u^{+}_{\mu}\partial^{\mu}$). The superfield $\Omega(z)$ in (5.11), (5.12) takes values in the YM gauge group and it is a functional of $A^{\mu}(z)$, solving the equation $(u^{+}_{\mu}\nabla^{\mu})\Omega = 0$:

$$\Omega(z) = P\exp\left\{-ig\int^{x^{-}} u^{+}_{\mu}A^{\mu}(x(y^{-}; u, v), \theta, u, v)\,\mathrm{d}y^{-}\right\}, \tag{5.13}$$

$$x^{-} \equiv u^{-}_{\mu}x^{\mu}, \qquad x^{\mu}(y^{-}; u, v) \equiv \left(\eta^{\mu\nu} + u^{+\mu}u^{-\nu}\right)x_{\nu} - u^{+\mu}y^{-}.$$

Now, eqs. (5.10)–(5.13) are easily recognized as the nonlinear (non-abelian) analogue of eqs. (3.37)–(3.40) related with the superfield wave function of the $D = 10$ $N = 1$ BS superparticle.

Let us now derive the nonlinear equations satisfied by $Y^{+\frac{1}{2}a}(z)$ (5.11), $B^{a}(z)$ (5.12), which are implied by the (nonlinear) system (5.1), (5.3)–(5.9). First of all we get:

$$(\hat{D}^{ab}, \hat{D}^{-+})\begin{bmatrix} Y^{+\frac{1}{2}c}(z) \\ B^{c}(z) \end{bmatrix} = 0, \tag{5.14}$$

where

$$\hat{D}^{ab} = D^{ab} + \begin{bmatrix} S^{ab} & 0 \\ 0 & V^{ab} \end{bmatrix};$$

$$\hat{D}^{-+} = D^{-+} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.15}$$

with S^{ab}, V^{ab} the same as in (3.16), (3.17). Therefore, $\phi(z)$ (5.10) is itself harmonic superfield with external overall $SO(8)\times SO(1,1)$ indices belonging to the space $\mathscr{H}$ (3.37).

Further, acting with $D^{\pm a}$ on both sides of eqs. (5.11), (5.12) and using eqs. (5.9), (5.13), (5.3), (5.6) together with certain algebra, we obtain the following equations for $\phi(z)$ (5.10) with explicitly separated linear and nonlinear parts:

$$D^{+a}\phi(z)=0, \tag{5.16}$$

$$\hat{D}^{-a}\phi(z)+\begin{bmatrix}[V_2^{-a}(\phi|z)]^{(Y)}\\ [V_2^{-a}(\phi|z)]^{(B)}\end{bmatrix}=0. \tag{5.17}$$

In (5.17), the linear operator $\hat{D}^{-a}$ is the same as in (3.29) (i.e. $\hat{D}^{-a}$ is the modified D^{-a} operator due to the harmonic "spin" part of $\hat{D}^{ab}$ and the non-zero SO(1,1) charge matrix of $\hat{D}^{-+}$ in (5.15). The nonlinear parts in (5.17) read:

$$\begin{aligned}[V_2^{-a}(\phi|z)]^{(Y)b}\equiv{}& ig\left[Y^{+\frac{1}{2}b},\frac{1}{\partial^+}B^a\right]\\ &+\tfrac{1}{2}ig\frac{1}{\partial^+}\left[Y^{+\frac{1}{2}b},B^a\right]+ig(S^{ac})^{bd}\frac{1}{\partial^+}\left[Y_d^{+\frac{1}{2}},B_c\right],\end{aligned} \tag{5.18}$$

$$[V_2^{-a}(\phi|z)]^{(B)b}\equiv gC^{ab}\frac{1}{(\partial^+)^2}\left(\left\{Y^{+\frac{1}{2}c},Y_c^{+\frac{1}{2}}\right\}-i\left[B_c,\partial^+B^c\right]\right)+ig\left[B^b,\frac{1}{\partial^+}B^a\right]. \tag{5.19}$$

In the course of derivation of eqs. (5.17) and below the following useful relation is used:

$$Y^{+\frac{1}{2}a}(z)=\left(v^{+\frac{1}{2}}\sigma^a\right)^{\alpha}\Omega^{-1}(z)W_\alpha(z)\Omega(z), \tag{5.20}$$

which is a consequence from (5.3) and (5.11).

The next step is to operate with D^α on both sides of eqs. (5.10), (5.12) and use eqs. (5.3)–(5.5) and (5.13), (5.20) to obtain (disentangling again linear and nonlinear parts):

$$\hat{D}^{\alpha}\phi(z)+\begin{bmatrix}[V_1^{\alpha}(\phi|z)]^{(Y)}\\ [V_1^{\alpha}(\phi|z)]^{(B)}\end{bmatrix}=0, \tag{5.21}$$

where $\hat{D}^\alpha$ is the linear operator defined in (3.27) and

$$\left[V_1^\alpha(\phi|z)\right]^{(Y)a} \equiv 2ig\left(v^{+\frac{1}{2}}\sigma^b\right)^\alpha\left\{\frac{1}{\partial^+}Y_b^{+\frac{1}{2}}, Y^{+\frac{1}{2}a}\right\} - \tfrac{1}{2}g\left(v^{+\frac{1}{2}}\sigma^a\sigma^{bc}\right)^\alpha\left[B_b, B_c\right]$$

$$-ig\left(v^{+\frac{1}{2}}\sigma^a\right)^\alpha\frac{1}{\partial^+}\left(\left\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\right\} - i\left[B_c, \partial^+ B^c\right]\right), \tag{5.22}$$

$$\left[V_1^\alpha(\phi|z)\right]^{(B)a} \equiv 2ig\left(v^{+\frac{1}{2}}\sigma^b\right)^\alpha\frac{1}{\partial^+}\left[\frac{1}{\partial^+}Y_b^{+\frac{1}{2}}, \partial^+ B^a\right]$$

$$-ig\left(v^{+\frac{1}{2}}\sigma^b\sigma^a\sigma^c\right)^\alpha\frac{1}{\partial^+}\left[B_c, Y_b^{+\frac{1}{2}}\right]. \tag{5.23}$$

Finally, from eqs. (5.7) and (5.6) and substituting eqs. (5.11), (5.12), (5.20) we get:

$$0 = \left(v^{+\frac{1}{2}}\sigma^a\right)^\alpha \Omega^{-1}\nabla\!\!\!/_{\alpha\beta}\left(\nabla\!\!\!/ W\right)^\beta\Omega$$

$$= \left(-\partial^2\right)Y^{+\frac{1}{2}a}(z) + \left[V_0(\phi|z)\right]^{(Y)a}, \tag{5.24}$$

$$0 = -u_\nu^a\Omega^{-1}\left(\nabla_\mu F^{\mu\nu} - gW\sigma^\nu W\right)\Omega$$

$$= \left(-\partial^2\right)B^a(z) + \left[V_0(\phi|z)\right]^{(B)a}, \tag{5.25}$$

where the nonlinear parts read:

$$\left[V_0(\phi|z)\right]^{(Y)a} \equiv -ig\left(\partial^b\left[B_b, Y^{+\frac{1}{2}a}\right] + \left[B_b, \nabla'^b Y^{+\frac{1}{2}a}\right] + \left[\partial^+ B_b, \frac{1}{\partial^+}\nabla'^b Y^{+\frac{1}{2}a}\right]\right)$$

$$+2ig\,\partial^+\left[\frac{1}{(\partial^+)^2}\left(\nabla_c'\,\partial^+ B^c - g\left\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\right\}\right), Y^{+\frac{1}{2}a}\right]$$

$$-2ig\left[\partial^+ B_c, \left(S^{cd}\right)^{ab}\frac{1}{\partial^+}\nabla_d' Y_b^{+\frac{1}{2}}\right] - ig\left[F_{cd}', \left(S^{cd}\right)^{ab}Y_b^{+\frac{1}{2}}\right]; \tag{5.26}$$

$$\left[V_0(\phi|z)\right]^{(B)a} \equiv \partial_b\left(\nabla'^a B^b\right) - \partial^+\nabla'^a\frac{1}{(\partial^+)^2}\left(\nabla_c'\,\partial^+ B^c - g\left\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\right\}\right)$$

$$+ig\left[\frac{1}{(\partial^+)^2}\left(\nabla_c'\,\partial^+ B^c - g\left\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\right\}\right), \partial^+ B^a\right]$$

$$+ig\left[B_b, F'^{ab}\right] + 2g\left(S^{ac}\right)_{bd}\left\{\frac{1}{\partial^+}\nabla_c' Y^{+\frac{1}{2}b}, Y^{+\frac{1}{2}d}\right\}$$

$$-g\left\{\frac{1}{\partial^+}\nabla'^a Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\right\} \tag{5.27}$$

with the following notations:

$$\nabla'^{a} \equiv \partial^{a} + ig[B^{a}, \cdot\,], \qquad F'^{ab} \equiv \partial^{a}B^{b} - \partial^{b}B^{a} + ig[B^{a}, B^{b}]$$

and $(S^{ab})_{cd}$ as in (3.17).

Thus, the nonlinear system (5.1), (5.3)–(5.7) of the Nilsson constraint equations and their consequences from the Bianchi identities together with (5.8), (5.9) implying the on-shell independence of $A^{\alpha}(x,\theta,u,v)$, $A^{\mu}(x,\theta,u,v)$ on the auxiliary harmonic variables (u,v) is reduced via the nonlinear field transformation (5.10)–(5.13) to the nonlinear system (5.16), (5.17), (5.21), (5.24), (5.25) for the harmonic superfields $\phi(z)$ (5.10):

$$D^{+a}\phi(z) = 0, \tag{5.28}$$

$$\hat{D}^{-a}\phi(z) + V_2^{-a}(\phi|z) = 0, \tag{5.29}$$

$$\hat{D}^{\alpha}\phi(z) + V_1^{\alpha}(\phi|z) = 0, \tag{5.30}$$

$$(-\partial^{2})\phi(z) + V_0(\phi|z) = 0, \tag{5.31}$$

with the nonlinear parts defined in (5.18), (5.19), (5.22), (5.23), (5.26), (5.27). (Since $\phi(z)$ (5.10) are harmonic superfields, (5.14) are *identically* satisfied.)

Now we shall establish the inverse statement, namely, starting from the nonlinear system (5.28)–(5.31) for the *harmonic* superfields $\phi(z)$ (5.10), we can exactly recover the original system (5.1), (5.3)–(5.7) in terms of the *ordinary* superfields $A^{\alpha}(x,\theta)$, $A^{\mu}(x,\theta)$. To this end we consider the following nonlinear field transformation

$$\phi(z) \equiv \begin{bmatrix} Y^{+\frac{1}{2}a}(z) \\ B^{a}(z) \end{bmatrix} \to \begin{bmatrix} A^{\alpha}(z) \\ A^{\mu}(z) \end{bmatrix}, \tag{5.32}$$

$$A^{\alpha}(z) = 2i\left(v^{+\frac{1}{2}}\sigma^{a}\right)^{\alpha}\hat{\Omega}(z)\left(\frac{1}{\partial^{+}}Y_a^{+\frac{1}{2}}(z)\right)\hat{\Omega}^{-1}(z) - \frac{1}{g}D^{\alpha}\hat{\Omega}(z)\hat{\Omega}^{-1}(z), \tag{5.33}$$

$$\begin{aligned} A^{\mu}(z) = {} & \hat{\Omega}(z)u_{\mu}^{a}B^{a}(z)\hat{\Omega}^{-1}(z) \\ & - u_{\mu}^{+}\hat{\Omega}\frac{1}{(\partial^{+})^{2}}\left[\nabla_c'\,\partial^{+}B^{c}(z) - g\left\{Y^{+\frac{1}{2}c}(z), Y_c^{+\frac{1}{2}}(z)\right\}\right]\hat{\Omega}^{-1}(z) \\ & + \frac{i}{g}\partial^{\mu}\hat{\Omega}(z)\,\hat{\Omega}^{-1}(z), \end{aligned} \tag{5.34}$$

where the harmonic superfield $\hat{\Omega}(z)$ takes values in the YM gauge group and it is functional of $B^a(z)$ defined by the equations:

$$D^{+a}\hat{\Omega} = 0, \tag{5.35}$$

$$\hat{\Omega}^{-1}D^{-a}\hat{\Omega} = -ig\frac{1}{\partial^+}B^a. \tag{5.36}$$

Note, that eqs. (5.17) (together with (5.19)) are the integrability conditions for the overdetermined system (5.36).

From the explicit form of (5.33), (5.34) it is seen that the new fields $A^\alpha(z)$, $A^\mu(z)$ are harmonic superfields (cf. (3.36), (2.11)), i.e. the equations:

$$(D^{ab}, D^{-+})\begin{bmatrix} A^\alpha(z) \\ A^\mu(z) \end{bmatrix} = 0 \tag{5.37}$$

are identically fulfilled.

First, applying the harmonic operators $D^{\pm a}$ (2.9) on $A^\alpha(z)$, $A^\mu(z)$ as defined by (5.33)–(5.36) and using (5.28), (5.29) and (5.19) we easily obtain:

$$D^{\pm a}\begin{bmatrix} A^\alpha(z) \\ A^\mu(z) \end{bmatrix} = 0, \tag{5.38}$$

which together with the identically fulfilled (5.37) yields the on-shell independence of $A^\alpha(z)$, $A^\mu(z)$ on (u, v):

$$A^\alpha = A^\alpha(x, \theta), \qquad A^\mu = A^\mu(x, \theta). \tag{5.39}$$

As a second step we consider the following expression:

$$\frac{1}{g}\left(\{\nabla^\alpha, \nabla^\beta\} - 2i\nabla\!\!\!/^{ab}\right) \equiv -\tfrac{1}{16}(\sigma_\mu)^{\alpha\beta}F^\mu - \frac{1}{32(5!)}(\sigma_{\mu_1\cdots\mu_5})^{\alpha\beta}F^{\mu_1\cdots\mu_5}, \tag{5.40}$$

where the covariant derivatives $\nabla^\alpha = D^\alpha + g[A^\alpha, \cdot\}$, $\nabla^\mu = \partial^\mu + ig[A^\mu, \cdot]$, are defined with the supergauge potentials from (5.33), (5.34). The coefficients of the σ-matrix expansion in (5.40) are (cf. [6]):

$$F^\mu \equiv (\sigma^\mu)_{\alpha\beta}\left(D^\alpha A^\beta + D^\beta A^\alpha + g\{A^\alpha, A^\beta\}\right) - 32A^\mu, \tag{5.41}$$

$$F^{\mu_1\cdots\mu_5} \equiv (\sigma^{\mu_1\cdots\mu_5})_{\alpha\beta}\left(D^\alpha A^\beta + D^\beta A^\alpha + g\{A^\alpha, A^\beta\}\right) \tag{5.42}$$

with A^α, A^μ from (5.33), (5.34).

According to (5.39) F^{μ} (5.41) and $F^{\mu_1\cdots\mu_5}$ (5.42) do not depend on the auxiliary harmonic variables (u, v), i.e.:

$$F^{\mu} = F^{\mu}(x,\theta), \qquad F^{\mu_1\cdots\mu_5} = F^{\mu_1\cdots\mu_5}(x,\theta). \tag{5.43}$$

Now, using the nonlinear definitions (5.33), (5.34) for A^{α}, A^{μ} and the obvious relation

$$\{D^{\alpha}, D^{\beta}\} = 2i\partial\!\!\!/^{\,ab},$$

one can easily show that

$$\hat{\Omega}^{-1}(z)\big(u^{+}_{\mu}F^{\mu}(x,\theta)\big)\hat{\Omega}(z) = 0,$$

$$\hat{\Omega}^{-1}(z)\big(u^{a_1}_{\mu_1}\dots u^{a_4}_{\mu_4}u^{+}_{\mu_5}F^{\mu_1\cdots\mu_5}(x,\theta)\big)\hat{\Omega}(z) = 0. \tag{5.44}$$

Since the harmonic coefficients u^{+}_{μ}, $u^{a_1}_{\mu_1}\dots u^{a_4}_{\mu_4}u^{+}_{\mu_5}$ in (5.44) are arbitrary and since F^{μ}, $F^{\mu_1\cdots\mu_5}$ do not depend on (u, v), (5.44) actually imply:

$$F^{\mu} = 0, \qquad F^{\mu_1\cdots\mu_5} = 0 \tag{5.45}$$

and, therefore, inserting (5.45) into (5.40):

$$F^{\alpha\beta} \equiv \frac{1}{g}\big(\{\nabla^{\alpha}, \nabla^{\beta}\} - 2i\nabla\!\!\!\!/^{\,\alpha\beta}\big) = 0,$$

which are exactly the original Nilsson constraint equations (5.1).

As a third step we introduce a harmonic superfield $W_{\alpha}(z)$ in the following way:

$$Y^{+\frac{1}{2}a}(z) = \left(v^{+\frac{1}{2}}\sigma^{a}\right)^{\alpha}\hat{\Omega}^{-1}(z)W_{\alpha}(z)\hat{\Omega}(z) \tag{5.46}$$

(i.e. $W_{\alpha}(z) = \hat{\Omega}(z)Y_{\alpha}(z)\hat{\Omega}^{-1}(z)$ in the notations (3.32)).

Now using eqs. (5.16)–(5.18) for $Y^{+\frac{1}{2}a}(z)$, we easily get:

$$D^{\pm a}W_{\alpha}(z) = 0, \tag{5.47}$$

which together with the identically fulfilled $(D^{ab}, D^{+-})W_{\alpha}(z) = 0$ implies, that W_{α} does not depend on (u, v):

$$W_{\alpha} = W_{\alpha}(x,\theta). \tag{5.48}$$

Inserting (5.46) into (5.33) we get a relation between A^α and W_α:

$$\hat{\Omega}^{-1}(\sigma^+ W)^\alpha \hat{\Omega} = i\,\partial^+\left(\hat{\Omega}^{-1}A^\alpha\hat{\Omega} + \frac{1}{g}\hat{\Omega}^{-1}D^\alpha\hat{\Omega}\right). \tag{5.49}$$

Using $u_\mu^+ A^\mu = ig\,\partial^+\hat{\Omega}\,\hat{\Omega}^{-1}$ (following from (5.34) by multiplying both sides with u_μ^+) we can rewrite (5.49) in the following form:

$$\hat{\Omega}^{-1}(z)u_\mu^+\left[F^{\alpha\mu}(x,\theta) - (\sigma^\mu)^{\alpha\beta}W_\beta(x,\theta)\right]\hat{\Omega}(z) = 0, \tag{5.50}$$

where

$$F^{\alpha\mu} \equiv D^\alpha A^\mu + i\,\partial^\mu A^\alpha + g[A^\alpha, A^\mu] \text{ (cf. notations (5.2))}.$$

Thus (5.50) actually imply:

$$F^{\alpha\mu}(x,\theta) - (\sigma^\mu W)^\alpha(x,\theta) = 0,$$

i.e. the original superfield eq. (5.3).

Now, having established properties (5.39), (5.48), i.e. the on-shell independence on (u, v) of A^α, A^μ, W_α defined in terms of $Y^{+\frac{1}{2}a}(z)$, $B^a(z)$ through (5.33), (5.34) and (5.46), it is straightforward to derive the following consequences of the nonlinear system (5.28)–(5.31) for $Y^{+\frac{1}{2}a}(z)$, $B^a(z)$:

$$\hat{\Omega}^{-1}(z)u_\mu^+u_\nu^a\left[\nabla^\alpha F^{\mu\nu}(x,\theta) - ((\sigma^\mu\nabla^\nu - \sigma^\nu\nabla^\mu)W)^\alpha(x,\theta)\right]\hat{\Omega}(z) = 0, \tag{5.51}$$

$$\hat{\Omega}^{-1}(z)(v^{+\frac{1}{2}}\sigma^a)^\beta\left[\nabla^\alpha W_\beta(x,\theta) + \tfrac{1}{2}i(\sigma^{\mu\nu})^\alpha_\beta F_{\mu\nu}(x,\theta)\right]\hat{\Omega}(z) = 0, \tag{5.52}$$

$$\hat{\Omega}^{-1}(z)u_\nu^+\left[\nabla_\mu F^{\mu\nu}(x,\theta) - gW\sigma^\nu W(x,\theta)\right]\hat{\Omega}(z) = 0, \tag{5.53}$$

$$\begin{aligned}
&\hat{\Omega}^{-1}(z)\tfrac{1}{2}(v^{+\frac{1}{2}}\sigma^b\sigma^a)_\alpha\left[(\nabla W)^\alpha(x,\theta)\right]\hat{\Omega}(z)\\
&= \partial^+\left[\left(D^{-a} - \frac{1}{2}\frac{\partial^a}{\partial^+}\right)Y^{+\frac{1}{2}b}(z) - \frac{\partial^c}{\partial^+}(S^{ac})^b_d Y^{+\frac{1}{2}d}(z)\right] + \partial^+\left[V_2^{-a}(\phi|z)\right]^{(Y)b}\\
&= 0 \qquad \text{(eq. (5.17))}
\end{aligned} \tag{5.54}$$

Once again, since the terms in the square brackets on the left-hand-sides of (5.51)–(5.54) do not depend on the harmonic variables (u, v), these equations imply the rest (5.4)–(5.7) of the nonlinear system for the ordinary superfields $A^\alpha(x,\theta)$, $A^\mu(x,\theta)$.

This finishes the proof of the equivalence between the Nilsson constraint equations (5.1) together with their consequences from the Bianchi identities (5.3)–(5.7) in terms of ordinary superfields $A^{\alpha}(x,\theta)$, $A^{\mu}(x,\theta)$ and the nonlinear system (5.28)–(5.31) in terms of harmonic superfields $Y^{+\frac{1}{2}a}(z)$, $B^{a}(z)$, where both sets of superfields are related through the nonlinear field transformation (5.10)–(5.13). Thus the system (5.28)–(5.31) provides alternatively the complete on-shell superspace description of $D=10$ $N=1$ SYM.

In particular one immediately notices that in the linearized case ($g=0$, i.e. $V_0(\phi|z)$, $V_1^{\alpha}(\phi|z)$, $V_2^{-a}(\phi|z)=0$) the harmonic superfield system (5.28)–(5.31) precisely reduces to the system of Dirac constraint equations (3.26)–(3.29) for the wave function $\phi(z)$ of the super-Poincaré covariantly quantized $D=10$ $N=1$ BS superparticle.

6. Off-shell superspace action for $D = 10$ SYM

In this section we shall review our general construction of action principle for arbitrary consistent overdetermined systems of nonlinear field equations [12] and, subsequently, shall apply it to derive a superspace action for $D=10$ SYM in terms of unconstrained (off-shell) superfields (cf. also [12]).

Let us consider the following general overdetermined system of $\mathcal{N}>1$ nonlinear equations:

$$\mathscr{L}_A(\phi|z) \equiv L_A\Phi(z) + V_A(\phi|z) = 0, \qquad A=1,\dots,\mathcal{N}, \tag{6.1}$$

$$V_A(\phi|z) \equiv \sum_{n\geqslant 0}\int \mathrm{d}z_1\dots\mathrm{d}z_{n+2}V_A^{(n+2)}(z;z_1,\dots,z_{n+2})\phi(z_1)\dots\phi(z_{n+2}). \tag{6.2}$$

In (6.1) the function $\phi(z)$ is defined on a (graded) linear space $\mathscr{R}$ and it takes values in another (graded) vector space $\mathscr{U}$, i.e. has a vector index $\phi=(\phi^a(z))$. Also, $\phi(z)$ is taken to be *real*. L_A are (graded) linear operators with at most second order derivatives and are, correspondingly, matrices ($L_A\equiv(L_A^{ab})$) in the vector space $\mathscr{U}$. Clearly, $V_A(\phi|z)=([V_A(\phi|z)]^a)$ are also vectors in $\mathscr{U}$. In the general discussion of this section the vector indices a, b will be suppressed for brevity.

Comparing (6.1) with (4.29) we see that the system (6.1) may be considered, from the point of view of second quantization as nonlinear generalization of the Dirac constraint equations for a first-quantized system with first-class Dirac constraints $\{L_A\}$, $A=1,\dots,\mathcal{N}$. Therefore the system (6.1) represents the nonlinear field equations of motion to be derived from an underlying field theory action which has to be a nonlinear generalization of (4.29)

The necessary conditions for consistency of the overdetermined system (6.1) are obtained by multiple application of antisymmetrized products of the linear operators L_B on $\mathscr{L}_A(\phi|z)$ (6.1) and by requiring the result to vanish when eqs. (6.1) are fulfilled. The first consistency condition:

$$L_A\mathscr{L}_B(\phi|z) + (-1)^{\epsilon_A\epsilon_B+1}L_B\mathscr{L}_{\mathscr{A}}(\phi|z) = 0$$

yields (cf. (4.25)) for the linear and nonlinear parts respectively:

$$[L_A, L_B\} \equiv L_AL_B + (-1)^{\epsilon_A\epsilon_B+1}L_BL_A = f_{AB}^{C}L_C, \tag{6.3}$$

$$\begin{aligned} & L_AV_B(\phi|z) + (-1)^{\epsilon_A\epsilon_B+1}L_BV_A(\phi|z) - f_{AB}^{C}V_C(\phi|z) \\ &= \int \mathrm{d}z' \left[\frac{\delta V_B(\phi|z)}{\delta\phi(z')}\mathscr{L}_A(\phi|z') + (-1)^{\epsilon_A\epsilon_B+1}\frac{\delta V_A(\phi|z)}{\delta\phi(z')}\mathscr{L}_B(\phi|z')\right] \end{aligned} \tag{6.4}$$

$$(=0 \text{ on the surface of equations } (6.1)).$$

In (6.3), (6.4) f_{AB}^{C} are in general linear operators and ϵ_A, ϵ_B are the Grassmann parities of L_A, L_B correspondingly. In eq. (6.4) the operators L_A act on $V_B(\phi|z)$ defined in eq. (6.2) as on functions of z. The next consistency condition

$$\left[L_C(-1)^{\epsilon_B+\epsilon_C}L_AV_B(\phi|z)\right]_{\text{antisymm}(A,B,C)} = 0 \text{ on the } (6.1) \text{ shell}$$

gives using (6.3), (6.4):

$$\left[f_{ABC}^{(2)DE}(-1)^{\epsilon_D}f_{AD}^{G}\right]_{\text{antisymm}(A,B,C)}V_G(\phi|z) = 0,$$

where the operator $f_{ABC}^{(2)DE}$ is defined by:

$$f_{ABC}^{(2)\ DE}L_E = \left((-1)^{\epsilon_D+\epsilon_B+1}\left\{(-1)^{\epsilon_D\epsilon_C}\left[f_{AB}^{D}, L_C\right] + f_{AB}^{G}f_{GC}^{D}\right\}\right)_{\text{antisymm}(ABC)} \tag{6.5}$$

and antisymmetrization means the same as in (4.37):

$$\mathscr{M}_{\dots BA\dots} = (-1)^{(\epsilon_A+1)(\epsilon_B+1)}\mathscr{M}_{\dots AB\dots}$$

For most interesting systems it turns out that:

$$f_{ABC}^{(2)\ DE} = 0. \tag{6.6}$$

Let us immediately note, that if the set of operators L_A is viewed as a first-quantized system of Dirac first class hamiltonian constraints (cf. (6.3) and (4.25)), then

$f^{(2)\,DE}_{ABC}$ defined by (6.5) is precisely the so called second order BFV structure function [11]. Its vanishing (6.6) means that the corresponding hamiltonian system is first-rank, i.e. the corresponding BRST charge does not possess higher order ghost terms, as in (4.26).

Our general construction of an action principle for the system (6.1) works under the following general assumptions:

(i) The number N_b of bosonic operators L_A in (6.1) (i.e. with $\epsilon_A = 0$) has to be *odd*;

(ii) The linear operators L_A must be functionally independent;

(iii) Condition (6.6) holds.

Condition (iii) means that the only nontrivial consistency conditions for the system (6.1) are given by (6.3), (6.4).

From the point of view of second quantization, conditions (ii) and (iii) mean that the underlying first-quantized system of Dirac first-class constraints $\{L_A\}$ is BFV-irreducible and first-rank.

Since the system (6.1) comprises $\mathcal{N} = N_b + N_f > 1$ matrix equations it is of course impossible to find an action functional $S = S[\phi]$, depending on $\phi(z)$ alone such that (6.1) would arise as equations of motion $\delta S/\delta\phi(z) = 0$.

Our general construction of an action principle for the overdetermined system (6.1) proceeds in the following series of steps.

The *first step* is to rewrite the overdetermined set (6.1) of $\mathcal{N}$ (matrix) equations as a single (matrix) equation in terms of a (vector valued) field $\Phi(z, \eta)$ depending on auxiliary variables collectively denoted by η. The original field $\phi(z)$ from (6.1) enters as:

$$\Phi(z,\eta) = \phi(z) + \tilde{\Phi}(z,\eta), \qquad \tilde{\Phi}(z,\eta) = \sum_{n\geqslant 1} \frac{1}{n!}\eta^{A_1}\dots\eta^{A_n}\phi_{A_1\dots A_n}(z). \quad (6.7)$$

To this end we take:

$$\eta = (\eta^A) = (c^i, \chi^\alpha) \quad i = 1,\dots,N_b, \qquad \alpha = 1,\dots,N_f, \qquad A = 1,\dots,\mathcal{N} = N_f + N_b \quad (6.8)$$

to be the ghost variables associated with L_A, i.e. having opposite Grassmann parity $\epsilon(\eta^A) = \epsilon_A + 1$. Since $\phi(z)$ was taken to be real, the ghost-haunted field $\Phi(z, \eta)$ is likewise *real*. Then (6.7) is exactly the ghost-haunted wave function (4.22) entering the BFV-BRST quantization (4.27)–(4.28).

The new single (matrix) equation for $\Phi(z,\eta)$ replacing the system (6.1) is of the following general form:

$$Q(\Phi|z,\eta) \equiv Q_0\Phi(z,\eta) + \mathscr{V}(\Phi|z,\eta) = 0 \tag{6.9}$$

$$\mathscr{V}(\Phi|z,\eta) \equiv \sum_{n\geqslant 0}\int \mathrm{d}z_1\,\mathrm{d}\eta_1\ldots\mathrm{d}z_{n+2}\,d\eta_{n+2}$$
$$\times\mathscr{V}^{(n+2)}(z,\eta;z_1,\eta_1,\ldots,z_{n+2},\eta_{n+2})\Phi(z_1,\eta_1)\ldots\Phi(z_{n+2},\eta_{n+2}). \tag{6.10}$$

The linear operator Q_0 entering (6.9) is the BRST charge [11] corresponding to the algebra (6.3):

$$Q_0 = \eta^A L_A + \tfrac{1}{2}(-1)^{\epsilon_B}\eta^B\eta^C f^A_{CB}\frac{\partial}{\partial\eta^A} \tag{6.11}$$

and $\mathscr{V}(\Phi, z, \eta)$ possesses the properties $(\delta(\eta) \equiv \prod_{A=1}^{\mathscr{N}}\delta(\eta^A))$:

$$\int \mathrm{d}\eta\,\delta(\eta)\mathscr{V}(\Phi|z,\eta) = 0, \qquad \int \mathrm{d}\eta\,\delta(\eta)\frac{\partial}{\partial\eta^A}\mathscr{V}(\Phi|z,\eta) = V_A(\phi|z). \tag{6.12a,b}$$

Eqs. (6.11), (6.12b) ensure that the single equation (6.9) for $\Phi(z,\eta)$ contains the original nonlinear system (6.1):

$$0 = \int \mathrm{d}\eta\,\delta(\eta)\frac{\partial}{\partial\eta^A}Q(\Phi|z,\eta) = L_A\phi(z) + V_A(\phi|z).$$

Let us point out that in each ghost integral *first* the integration over the fermionic ghosts c^i (6.8) is performed:

$$\int \mathrm{d}\eta\,\mathscr{F}(\eta) = \int \mathrm{d}\chi\left[\int \mathrm{d}c\,\mathscr{F}(c,\eta)\right], \qquad \int \mathrm{d}c\,c^{i_1}\ldots c^{i_M} = \delta_{MN_b}\epsilon^{i_1\cdots i_{N_b}} \tag{6.13}$$

Clearly, (6.11) tells us that (6.9) is precisely the appropriate nonlinear generalization of the BFV equation (4.27), i.e. the BFV physical state condition.

The *second step* is to find the gauge invariance exhibited by the new single equation (6.9) such that the equations of motion implied by (6.9) for the "non-physical" part $\tilde{\Phi}(z,\eta)$ of the ghost-haunted field $\Phi(z,\eta)$ (6.7) should have pure-gauge solutions, whereas eqs. (6.1) for the original field $\phi(z)$ should be gauge-invariant. This gauge symmetry must yield the appropriate nonlinear generalizations of (4.28), (4.32), (4.34), (4.36).

The required gauge invariance has the form:

$$\delta_\Lambda \Phi(z,\eta) = \int \mathrm{d}z'\,\mathrm{d}\eta'\,\Lambda(z',\eta')\frac{\delta Q(\Phi|z,\eta)}{\delta\Phi(z',\eta')}$$

$$= Q_0\Lambda(z,\eta) + \int \mathrm{d}z'\,\mathrm{d}\eta'\,\Lambda(z',\eta')\frac{\delta \mathscr{V}(\Phi|z,\eta)}{\delta\Phi(z',\eta')} \tag{6.14}$$

and the gauge invariance of (6.9) under (6.14) implies*:

$$\int \mathrm{d}z'\,\mathrm{d}\eta'\,Q(\Phi|z',\eta')\frac{\delta Q(\Phi|z,\eta)}{\delta\Phi(z',\eta')} = 0\,. \tag{6.15}$$

Inserting in (6.15) the expansion (6.9) for $Q(\Phi|z',\eta')$ one gets:

$$Q_0^2 = 0$$

(i.e. Q_0 is a nilpotent operator which is true by construction, see eqs. (6.11), (6.6)), and

$$Q_0\mathscr{V}(\Phi|z,\eta) + \int \mathrm{d}z'\,\mathrm{d}\eta'\left[Q_0\Phi(z',\eta') + \mathscr{V}(\Phi|z',\eta')\right]\frac{\delta \mathscr{V}(\Phi|z,\eta)}{\delta\Phi(z',\eta')} = 0\,. \tag{6.16}$$

Therefore, it is natural to call eq. (6.15) the nonlinear nilpotency condition.

Also note, that due to (6.12a), the original field $\phi(z)$ is *inert* under the gauge transformation (6.14):

$$\delta_\Lambda\phi(z) = \int \mathrm{d}\eta\,\delta(\eta)\,\delta_\Lambda\Phi(z,\eta)$$

$$= \int \mathrm{d}z'\,\mathrm{d}\eta'\,\Lambda(z',\eta')\frac{\delta}{\delta\Phi(z',\eta')}\left[\int \mathrm{d}\eta\,\delta(\eta) Q(\Phi|z,\eta)\right] \equiv 0$$

exactly as in the linear case (4.32).

* In fact, due to (6.15), the gauge parameter $\Lambda(z,\eta)$ in (6.14) is defined itself only modulo nonlinear transformations

$$\Lambda(z,\eta) \sim \Lambda(z,\eta) + \int \mathrm{d}z'\,\mathrm{d}\eta'\,\Lambda'(z',\eta')\frac{\delta Q(\Phi|z,\eta)}{\delta\Phi(z',\eta')}\,.$$

The *third step* is to derive the action, invariant under (6.14) and producing (6.9) as equation of motion. It is easily found to be:

$$S = \int \mathrm{d}z\,\mathrm{d}\eta\,\hat{H}\Phi(z,\eta)\bar{Q}(\Phi|z,\eta)$$

$$= \tfrac{1}{2}\int \mathrm{d}z\,\mathrm{d}\eta\,\hat{H}\Phi(z,\eta)Q_0(\Phi|z,\eta) + \int \mathrm{d}z\,\mathrm{d}\eta\,\hat{H}\Phi(z,\eta)\bar{\mathscr{V}}(\Phi|z,\eta) \quad (6.17)$$

with notations explained as follows. The linear operator $\hat{H}$ is defined to fulfill (T denotes operator transposition):

$$\hat{H}^{\mathrm{T}} = \hat{H}, \qquad Q_0^{\mathrm{T}}\hat{H} = \hat{H}Q_0. \quad (6.18)$$

A typical form of $\hat{H}$ is $\hat{H}\Phi(z,\eta) = R\Phi(\rho_1 z, \rho_2 z)$ where R is a matrix acting on the vector-valued field, $\rho_{1,2}$ are numbers taking the values $\pm 1, \pm i$ (cf. (4.42), (4.43)). Let us recall that, since $\Phi(z,\eta)$ is real, the free part of the action (6.17) is bilinear (instead of hermitean) form in Φ. The functional $\bar{Q}(\Phi|z,\eta)$ is defined through the relation:

$$\left[1 + \int \mathrm{d}z'\,\mathrm{d}\eta'\,\Phi(z',\eta')\frac{\delta}{\delta\Phi(z',\eta')}\right]\bar{Q}(\Phi|z,\eta) = Q(\Phi|z,\eta) \quad (6.19)$$

which simply means:

$$\bar{Q}(\Phi|z,\eta) = \tfrac{1}{2}Q_0\Phi(z,\eta) + \bar{\mathscr{V}}(\Phi|z,\eta), \quad (6.20)$$

where $\bar{\mathscr{V}}(\Phi|z,\eta)$ is given by a series of the same form as for $\mathscr{V}(\Phi|z,\eta)$ (6.10) with additional multiplication of each $\mathscr{V}^{(n+2)}$ by the factor $(n+3)^{-1}$:

$$\bar{\mathscr{V}}(\Phi|z,\eta) = \sum_{n\geqslant 0}\frac{1}{n+3}\int \mathrm{d}z_1\,\mathrm{d}\eta_1\ldots\mathrm{d}z_{n+2}\,\mathrm{d}\eta_{n+2}$$

$$\times\mathscr{V}^{(n+2)}(z,\eta;z_1,\eta_1\ldots,z_{n+2},\eta_{n+2})\Phi(z_1,\eta_1)\ldots\Phi(z_{n+2},\eta_{n+2}). \quad (6.21)$$

Since $\bar{Q}(\Phi|z,\eta)$ (6.20)–(6.21) enters the action functional (6.17) where one can freely symmetrize the fields $\Phi(z,\eta)$ entering in the various terms, we immediately find that $\bar{Q}$ (6.20) or, equivalently, Q (6.9) should satisfy the antisymmetry condition:

$$\frac{\delta \hat{H}Q(\Phi|z,\eta)}{\delta\Phi(z',\eta')} = -\frac{\delta \hat{H}Q(\Phi|z',\eta')}{\delta\Phi(z,\eta)}\,. \tag{6.22}$$

The minus sign in (6.22) is due to the anticommutativity of the ghost measures (recall $N_b \equiv$ number of $c^i =$ odd):

$$\int \mathrm{d}\chi\,\mathrm{d}c \int \mathrm{d}\chi'\,\mathrm{d}c' = -\int \mathrm{d}\chi'\,\mathrm{d}c' \int \mathrm{d}\chi\,\mathrm{d}c$$

Now, it is straightforward to show that the action (6.17) is indeed invariant under the gauge transformation (6.14) provided the nonlinear nilpotency (6.15) and the antisymmetry condition (6.22) hold. Clearly the action (6.17) is precisely the nonlinear generalization of the free BFV-BRST action (4.40).

The *final step* is to derive the explicit expression of $\mathscr{V}(\Phi|z,\eta)$ (6.10) such that (6.15), (6.22) and (6.12) are satisfied. Using (6.11) and the consistency conditions (6.4) and inserting them into eq. (6.16) we find:

$$\begin{aligned}\mathscr{V}(\Phi|z,\eta) &= \eta^A V_A(\Phi(\cdot,\eta)|z)\\ &= \sum_{n\geqslant 0} \int \mathrm{d}z_1\ldots\mathrm{d}z_{n+2}\,\eta^A V_A^{(n+2)}(z;z_1,\ldots,z_{n+2})\Phi(z_1,\eta)\ldots\Phi(z_{n+2},\eta)\end{aligned} \tag{6.23}$$

and similarly:

$$\begin{aligned}&\bar{\mathscr{V}}(\Phi|z,\eta)\\ &= \sum_{n\geqslant 0} \frac{1}{n+3} \int \mathrm{d}z_1\ldots\mathrm{d}z_{n+2}\,\eta^A V_A^{(n+2)}(z;z_1,\ldots,z_{n+2})\Phi(z_1,\eta)\ldots\Phi(z_{n+2},\eta),\end{aligned} \tag{6.24}$$

where the kernels $V_A^{(n+2)}$ are exactly the same as in (6.2).

Eqs. (6.23), (6.24) are the principal result in the present general construction since now each object Q_0 (6.11), $\bar{\mathscr{V}}$ (6.21), (and similarly $\mathscr{V}$(6.10)) entering the action (6.17) is explicitly expressed in terms of objects $\{L_A\}, \{V_A^{(n+2)}\}$ entering the original nonlinear system (6.1), (6.2).

Let us now apply our general action principle to construct an off-shell superspace action for $D = 10$ SYM. From the mathematical point of view, the system (5.1), (5.3)–(5.7) is a consistent overdetermined system of nonlinear equations for the supergauge potentials $A^{\alpha}(x,\theta)$, $A^{\mu}(x,\theta)$. However, one can easily show that it cannot be written in the form (6.1) with Lorentz-covariant and independent linear operators L_A. On the other hand, it was shown in detail in sect. 5, that the nonlinear system (5.1), (5.3)–(5.7) is equivalent to the nonlinear system (5.28)–(5.31) in terms of the harmonic superfield

$$\phi(z) \equiv \begin{bmatrix} Y^{+\frac{1}{2}a}(z) \\ B^{a}(z) \end{bmatrix},$$

which is related to A^{α}, A^{μ} from (5.1), (5.3)–(5.7) through the nonlinear field transformation (5.10)–(5.13). Therefore, the harmonic superfield representation (5.28)–(5.31) of the $D=10$ SYM on-shell equations (5.1), (5.3)–(5.7) is a consistent overdetermined system of nonlinear field equations fulfilling all conditions (i), (ii), (iii) above for our action principle to work. Indeed:

(i) The number of bosonic operators L_A: $(-\partial)^2$, D^{+a}, $\hat{D}^{-a}$ from (5.28)–(5.31) is odd ($=17$);

(ii) All linear operators $\{L_A\} \equiv \{(-\partial)^2, \hat{D}^{\alpha}, D^{+a}, \hat{D}^{-a}\}$ in (5.28)–(5.31) are BFV-irreducible, i.e. functionally independent;

(iii) The set of $\{L_A\}$ is first-rank, i.e. the second order BFV structure function vanishes (6.6).

Thus our action principle (eq. (6.17)) yields the following superspace action in terms of off-shell unconstrained superfields for $D=10$ SYM:

$$\begin{aligned} S_{\mathrm{SYM}} = {} & \tfrac{1}{2}\int \mathrm{d}z\,\mathrm{d}\eta\, \hat{H}\Phi(z,\eta) Q_0 \Phi(z,\eta) \\ & + \int \mathrm{d}z\,\mathrm{d}\eta\, \hat{H}\Phi(z,\eta)\big[cV_0(\Phi(\cdot,\eta)|z) + \chi_{\alpha}V_1^{\alpha}(\Phi(\cdot,\eta)|z) \\ & \qquad + \eta_a^{+} V_2^{-a}(\Phi(\cdot,\eta)|z)\big] \end{aligned} \tag{6.25}$$

with the notations:

$$\begin{aligned} z &\equiv \left(x^{\mu}, \theta_{\alpha}, u_{\mu}^{a}, v_{\alpha}^{\pm\frac{1}{2}}\right), \\ \mathrm{d}z &\equiv (\mathrm{d}^{10}x^{\mu})(\mathrm{d}^{16}\theta_{\alpha})(\mathrm{d}^{80}u_{\mu}^{a})(\mathrm{d}^{32}v_{\alpha}^{\pm\frac{1}{2}}) \\ & \quad \times \prod_{a,b}\delta\left(u_{\mu}^{a}u^{b\mu} - C^{ab}\right)\prod_{a,\pm}\delta\left(u_{\mu}^{a}v^{\pm\frac{1}{2}}\sigma^{\mu}v^{\pm\frac{1}{2}}\right)\delta\left(\left(v^{+\frac{1}{2}}\sigma_{\mu}v^{+\frac{1}{2}}\right)\left(v^{-\frac{1}{2}}\sigma^{\mu}v^{-\frac{1}{2}}\right)+1\right); \end{aligned} \tag{6.26}$$

$$\eta \equiv (\eta^{A}) \equiv \left(c, \chi_{\alpha}, \eta^{\pm a}\right), \qquad \mathrm{d}\eta \equiv \mathrm{d}c(\mathrm{d}^{16}\chi_{\alpha})(\mathrm{d}^{8}\eta^{+a})(\mathrm{d}^{8}\eta^{-a}), \tag{6.27}$$

$$\Phi(z,\eta) \equiv \begin{bmatrix} \mathscr{Y}^{+\frac{1}{2}a}(z,\eta) \\ \mathscr{B}^{a}(z,\eta) \end{bmatrix} \equiv \begin{bmatrix} \left(v^{+\frac{1}{2}}\sigma^{a}\right)^{\alpha}\mathscr{Y}_{\alpha}(z,\eta) \\ u_{\mu}^{a}\mathscr{B}^{\mu}(z,\eta) \end{bmatrix}, \tag{6.28}$$

where $\mathscr{Y}_\alpha(z,\eta)$, $\mathscr{B}^\mu(z,\eta)$ are ghost haunted superfields without external $SO(8)\times SO(1,1)$ indices (cf. (4.11), (4.15), (4.16)) and the functionals

$$V_0(\Phi(\cdot,\eta)|z),\, V_1^\alpha(\Phi(\cdot,\eta)|z),\, V_2^{-a}(\Phi(\cdot,\eta)|z)$$

in the interacting part of $S_{\rm SYM}$ (6.25) are exactly the same as (5.26), (5.27), (5.22), (5.23), (5.18), (5.19), where the usual real harmonic superfields $Y^{+\frac{1}{2}a}(z)$, $B^a(z)$ are substituted with the corresponding real ghost-haunted harmonic superfields $\mathscr{Y}^{+\frac{1}{2}a}(z,\eta)$, $\mathscr{B}^a(z,\eta)$ (6.28).

The way the supergauge potentials $A^\alpha(x,\theta)$, $A^\mu(x,\theta)$ of $D=10$ SYM enter in the action (6.25) is given by the following nonlinear ghost-haunted superfield transformation:

$$\begin{bmatrix} \mathscr{Y}^{+\frac{1}{2}a}(z,\eta) \\ \mathscr{B}^a(z,\eta) \end{bmatrix} \quad (\text{eq. } (6.28)) \to \begin{bmatrix} \mathscr{A}^\alpha(z,\eta) \\ \mathscr{A}^\mu(z,\eta) \end{bmatrix}, \tag{6.29}$$

$$\Phi(z,\eta)$$
$$\equiv \begin{bmatrix} \frac{1}{2}i(v^{+\frac{1}{2}}\sigma^a\sigma^-)_\alpha\partial^+\left[\Omega^{-1}(z,\eta)\mathscr{A}^\alpha(z,\eta)\Omega(z,\eta)+\frac{1}{g}\Omega^{-1}(z,\eta)D^\alpha\Omega(z,\eta)\right] \\ u_\mu^a\left[\Omega^{-1}(z,\eta)\mathscr{A}^\mu(z,\eta)\Omega(z,\eta)-\frac{i}{g}\Omega^{-1}(z,\eta)\,\partial^\mu\Omega(z,\eta)\right] \end{bmatrix}, \tag{6.30}$$

where $\Omega(z,\eta)$ is a functional of $\mathscr{A}^\mu(z,\eta)$ taking values in the YM gauge group and it is defined in complete analogy with (5.13):

$$\Omega(z) = P\exp\left\{-ig\int^{x^-} u_\mu^+\mathscr{A}^\mu(x(y^-;u,v),\theta,u,v;\eta)\,\mathrm{d}y^-\right\}, \tag{6.31}$$

$x^-\equiv u_\mu^- x^\mu$, $x^\mu(y^-;u,v)\equiv(\eta^{\mu\nu}+u^{+\mu}u^{-\nu})x_\nu - u^{+\mu}y^-$. Thus, the zeroth order term in the ghost expansion of $\Phi(z,\eta)$ (6.29)–(6.31) exactly coincides with the harmonic SYM superfield $\phi(z)$ (5.10)–(5.13), and, therefore, the usual SYM supergauge potentials $A^\alpha(x,\theta)$, $A^\mu(x,\theta)$ are identified as the *harmonic* (u,v) *independent* parts of the zeroth order terms in the ghost expansions (4.23) of $\mathscr{A}^\alpha(z,\eta)$ $\mathscr{A}^\mu(z,\eta)$ from (6.30) exactly as in the linearized case (sect. 4).

As a final remark, let us stress that the superspace action (6.25) is also manifestly invariant under the superspace YM gauge transformation of the ghost-haunted superfields $\mathscr{A}^{\mu}(z,\eta)$, $\mathscr{A}^{\alpha}(z,\eta)$:

$$\mathscr{A}^{\mu}(z,\eta) \to (\mathscr{A}^{\omega})^{\mu}(z,\eta) = \omega^{-1}(z,\eta)\left(\mathscr{A}^{\mu}(z,\eta) - \frac{i}{g}\partial^{\mu}\right)\omega(z,\eta),$$

$$\mathscr{A}^{\alpha}(z,\eta) \to (\mathscr{A}^{\omega})^{\alpha}(z,\eta) = \omega^{-1}(z,\eta)\left(\mathscr{A}^{\alpha}(z,\eta) + \frac{1}{g}D^{\alpha}\right)\omega(z,\eta). \quad (6.32)$$

This is because the action (6.25) depends on $\mathscr{A}^{\alpha}(z,\eta)$, $\mathscr{A}^{\mu}(z,\eta)$ only through the ghost-haunted superfield expression $\Phi(z,\eta)$ (6.30) which is itself invariant under (6.32).

Let us recapitulate the results of this section. We described here a general construction [12] of an off-shell action principle for arbitrary consistent overdetermined systems of nonlinear field equations. The main tool is the BFV-BRST ghost formalism [11]. The action (6.17) resembles the Siegel-Zwiebach-Witten-Neveu-West [36] construction of (super)string field actions but does not involve the peculiarities (star products, Chern-Simons forms etc.) specific to the field theory of the Ramond-Neveu-Schwarz (RNS) (super)string.

The main application presented here is the construction of a superspace action (6.25) for $D=10$ $N=1$ SYM in terms of unconstrained (off-shell) superfields. Because of bosonic variables (u, v, η^{A}), these superfields contain an infinite number of pure-gauge and auxiliary fields which are eliminated through the Witten-type nonlinear BFV gauge invariance (6.14) and through the usual superspace YM gauge invariance (6.33) of our superspace action. Let us particularly stress that, in our formalism, the YM gauge invariance (6.33) is *not* a part of the Witten-type gauge invariance (6.14) but it is an independent symmetry of our action (6.25). This phenomenon is most easily understood in the context of the heterotic GS superstring. Already its zero-mode (point-particle) limit contains the gauge invariant SYM whereas in the RNS formalism the YM gauge invariance arises *from* Witten's gauge invariance at the first excited string level in the NS sector.

7. Conclusions and outlook

The main objectives of the present paper may be summarized as follows.

(i) We describe in a pedagogical way the main ideas and concepts in the harmonic superstring program aimed at a consistent manifestly super-Poincaré covariant quantization of space-time supersymmetric strings (the GS superstrings).

The first crucial step is introduction of auxiliary harmonic variables allowing covariant disentangling of local fermionic gauge-invariances of the superstring. The next crucial step is the introduction of additional fermionic string coordinates enabling us to convert the set of mixed first- and second-class Dirac hamiltonian constraints of the GS superstring into a set of super-Poincaré covariant, functionally independent (BFV-irreducible) first-class constraints only.

This is inevitable in order to preserve manifest supersymmetry (Dirac brackets due to the second-class constraints would ruin the superspace geometry by causing the superstring coordinates x^{μ}, θ_{α} *not* to commute among themselves).

The introduction of the auxiliary harmonic and fermionic string variables is accompanied by introduction of appropriate additional gauge invariances beyond those of the GS superstring such that the new system (called harmonic GS superstring) is physically equivalent to the original GS model. We also made contact with a more recent formulation [24] extending the harmonic superstring program from the canonical hamiltonian formalism to the lagrangian functional-integral quantization formalism.

(ii) The effectiveness of the harmonic superstring program was further explicitly demonstrated by providing the full first-quantization analysis of the zero-mode (point particle) limit of the GS superstring – the $D = 10$ ($N = 1$) BS superparticle. The main result here is the derivation of the linearized Nilsson curvature constraints for $D = 10$ SYM *from* and establishing their equivalence to the manifestly super-Poincaré covariant Dirac constraint equations for the $D = 10$ $N = 1$ BS superparticle.

(iii) The preceding result was further generalized to the full nonlinear case by deriving a harmonic superfield representation of the nonlinear Nilsson constraints of $D = 10$ SYM reducing in the linearized case to the system of Dirac constraint equations for the $D = 10$ $N = 1$ BS superparticle.

(iv) We described the main steps of our construction of a covariant action principle for a very broad class of consistent overdetermined systems of nonlinear field equations. The only conditions for their structure are the following. The linear parts of the equations are identified as a system of quantized Dirac first-class constraints belonging to an underlying particle-like (or string-like) system which are BFV-irreducible and first rank (i.e. the second and higher BFV structure functions vanish and the corresponding BRST charge does not exhibit neither higher ghost terms nor ghosts for ghosts). In particular, a system of consistency equations on the interacting parts of the above nonlinear equations was formulated (eq. (6.4)) which allows in principle to find interacting (nonlinear) modifications of Dirac constraint equations for particle-like and string-like systems, i.e. to find the corresponding interacting field theoretic equations of motion.

(v) Our general action principle for overdetermined systems of nonlinear field equations was applied to derive a superspace action for $D = 10$ SYM in terms of unconstrained off-shell superfields, starting from the harmonic superfield representation of the Nilsson curvature constraints for $D = 10$ SYM. Thus, a solution was found to the long standing problem of an off-shell superspace formulation of $D = 10$ SYM. The same formalism can be applied to $D = 4$ $N = 4$ SYM and similar supersymmetric gauge theories which are formulated in terms of geometrical constraints on some of the relevant curvatures.

Although the $D = 10$ SYM action (6.25) is manifestly off-shell supersymmetric, this is at the price of having covariant nonlocal factors $(\partial^+)^{-1}$ (recall $\partial^+ \equiv v^{+\frac{1}{2}} \not\partial v^{+\frac{1}{2}}$). One may hope that by combining the present approach of section 6 with the formalism developed in [37] one will be able by further appropriate nonlinear field transformations of $\Phi(z, \eta)$ (6.28) to eliminate the nonlocality $(\partial^+)^{-1}$ factors.

The next most ambitious task is to apply the formalism presented in this paper to attack the issue of a manifestly super-Poincaré covariant field theory of the GS superstrings. The main problem here will be to find solutions for the field-theoretic superstring vertices coming from the string generalization of the consistency eqs. (6.4).

Appendix

GENERAL HARMONIC SUPERFIELDS AND PURE-GAUGE NATURE OF THE HARMONIC VARIABLES

In the $D = 4$ harmonic superspace approach [20] harmonic superfields are defined as functions on the extended N-superspace $z = (x^\mu, \theta^i_\alpha; u)$, $(i = 1, \ldots, N)$ where the variables u belong to a compact homogenous space G/H.

For $N = 2, 3$, G is the group of automorphisms of the extended super-Poincaré algebra $\mathrm{G} = \mathrm{SU}(N)$, whereas $\mathrm{H} = [\mathrm{U}(1)]^{N-1}$ [20].

In the present $D = 10$ case the appropriate homogenous space $\mathscr{L}/\mathrm{SO}(8) \times \mathrm{SO}(1,1)$ is noncompact, since the analog of the group-space G is here the space $\mathscr{L}$ defined by the kinematical constraints (2.4) on $u^a_\mu, v^{\pm\frac{1}{2}}_\alpha$. $\mathrm{H} = \mathrm{SO}(8) \times \mathrm{SO}(1,1)$ is the internal group of local rotations of $u^a_\mu, v^{\pm\frac{1}{2}}_\alpha$ (2.4). The fact that our harmonic superfields $\phi(z)$, $z \equiv (x^\mu, \theta_\alpha, u^a_\mu, v^{\pm\frac{1}{2}}_\alpha)$ are actually functions on $\mathscr{L}/\mathrm{SO}(8) \times \mathrm{SO}(1,1)$ is expressed by the property that they *identically* satisfy:

$$\hat{D}^{ab}\phi(z) \equiv (D^{ab} + \Sigma^{ab})\phi(z) = 0, \tag{C.1}$$

$$\hat{D}^{-+}\phi(z) \equiv (D^{-+} - \hat{q})\phi(z) = 0. \tag{C.2}$$

In (C.1), (C.2) D^{ab}, D^{+-} are the same as in (2.7), (2.8), i.e. they are "orbital" parts of the SO(8) and SO(1,1) rotations, whereas Σ^{ab} denotes the "spin" part of SO(8) and $\hat{q}$ denotes the SO(1,1) charge matrix.

In general, $\phi(z)$ may be a direct sum of components transforming under different inequivalent representations of the "spin"-part Σ^{ab} in (C.1) and possessing different half-integer or integer SO(1,1) charges in (C.2).

This is precisely the case in the present formalism – see eqs. (3.25), (3.30), (3.31). Therefore, it is sufficient to analyze (C.1), (C.2) for harmonic superfields of the form:

$$
\begin{aligned}
&\phi(z) \equiv \phi^{(q)\{c\}\{+\frac{1}{2}d\}\{-\frac{1}{2}e\}}(z), \\
&\{c\} \equiv (c_1, \dots, c_l), \qquad \{+\tfrac{1}{2}d\} \equiv (+\tfrac{1}{2}d_1, \dots, +\tfrac{1}{2}d_m), \\
&\{-\tfrac{1}{2}e\} \equiv (-\tfrac{1}{2}e_1, \dots, -\tfrac{1}{2}e_n),
\end{aligned}
\tag{C.3}
$$

with an overall SO(1,1) charge $q + \frac{1}{2}(m-n)$ (q is integer) and whose external SO(8) indices $(c_1, \dots, c_l), (d_1, \dots, d_m), (e_1, \dots, e_n)$ transform respectively under the harmonic (v), (s) and (c) representations [10]:

$$
\left[D^{ab} + \sum_{i=1}^{l} V^{ab}(i) + \sum_{j=1}^{m} S^{ab}(j) + \sum_{k=1}^{n} \tilde{S}^{ab}(k) \right] \phi(z) = 0, \tag{C.4}
$$

$$
\left[D^{-+} - \left(q + \tfrac{1}{2}m - \tfrac{1}{2}n \right) \right] \phi(z) = 0, \tag{C.5}
$$

In (C.4) $V^{ab}(i)$ denotes the action of V^{ab} [10] on the ith index c_i:

$$
V^{ab}(i)\phi(z) \equiv \left(V^{ab}\right)^{c_i}{}_{c_i'} \phi^{(q)(c_1, \dots, c_i', \dots, c_l)\{+\frac{1}{2}d\}\{-\frac{1}{2}e\}}
$$

and similarly for $S^{ab}(j)$, $\tilde{S}^{ab}(k)$.

Now, recalling the action of D^{ab} on $v^{\pm\frac{1}{2}\sigma^c}$ and u_μ^c [10] we find that each external SO(8) × SO(1,1) index c_i, $(+\frac{1}{2}d_j)$, $(-\frac{1}{2}e_k)$ of $\phi(z)$ (C.3) can be unambiguously saturated by $u_\mu^{c_i}$, $(v^{+\frac{1}{2}\sigma^{d_j}})^{\alpha_j}$, $(v^{-\frac{1}{2}\sigma^{e_k}})^{\beta_k}$. On the other hand the integer charge q can be unambiguously saturated by q vectors $u_\mu^{\pm} \equiv v^{\pm\frac{1}{2}}\sigma_\mu v^{\pm\frac{1}{2}}$ (depending on the sign of q). Therefore:

$$
\begin{aligned}
\phi(z)\ (\text{eq. (C.3)}) &= u_{\mu_1}^{\pm} \dots u_{\mu_q}^{\pm} u_{\nu_1}^{c_1} \dots u_{\nu_l}^{c_l} \times \left(v^{+\frac{1}{2}\sigma^{d_1}}\right)^{\alpha_1} \dots \left(v^{+\frac{1}{2}\sigma^{d_m}}\right)^{\alpha_m} \\
&\times \left(v^{-\frac{1}{2}\sigma^{e_1}}\right)^{\beta_1} \dots \left(v^{-\frac{1}{2}\sigma^{e_n}}\right)^{\beta_n} \phi^{\{\mu\}\{\nu\}}_{\{\alpha\}\{\beta\}}(z), \\
&\{\mu\} \equiv (\mu_1, \dots, \mu_q), \qquad \{\nu\} \equiv (\nu_1, \dots, \nu_l), \\
&\{\alpha\} \equiv (\alpha_1, \dots, \alpha_m), \qquad \{\beta\} \equiv (\beta_1, \dots, \beta_n),
\end{aligned}
\tag{C.6}
$$

where the coefficients superfields *identically* satisfy:

$$
\left(D^{ab}, D^{-+}\right) \phi^{\{\mu\}\{\nu\}}_{\{\alpha\}\{\beta\}}(z) = 0,
$$

i.e. they belong to the space $\mathscr{H}_0$ (3.36), (2.11) of harmonic superfields without any external SO(8) × SO(1,1) indices. By this we mean fields of the form:

$$\phi^{\{\mu\}\{\nu\}}_{\{\alpha\}\{\beta\}}(z) \equiv \sum_{\{\kappa\}\{\lambda\}\{\rho\}\{\tau\}} \left(\frac{u^+_{\kappa_1}}{p^+}\right)\cdots\left(\frac{u^+_{\kappa_r}}{p^+}\right)\left(\frac{u^-_{\lambda_1}}{p^-}\right)\cdots\left(\frac{u^-_{\lambda_s}}{p^-}\right)$$

$$\times u^+_{\rho_1}\cdots u^+_{\rho d}N \times u^-_{\tau_1}\cdots u^-_{\tau_N}\phi^{\{\mu\}\{\nu\}\{\kappa\}\{\lambda\}\{\rho\}\{\tau\}}_{\{\alpha\}\{\beta\}}(p,\theta). \qquad \text{(C.7)}$$

Here the coefficient (ordinary) superfields $\phi^{\{\mu\}\{\nu\}\{\kappa\}\{\lambda\}\{\rho\}\{\tau\}}_{\{\alpha\}\{\beta\}}(p,\theta)$ satisfy the following irreducibility properties which guarantee the uniqueness of the harmonic expansion (C.6), (C.7):

(i) $\phi^{\{\mu\}\{\nu\}\{\kappa\}\{\lambda\}\{\rho\}\{\tau\}}_{\{\alpha\}\{\beta\}}(p,\theta)$ are symmetric and traceless with respect to all Lorentz-vector indices.

(ii) They are transverse with respect to $\{\kappa\},\{\lambda\}$, and also they are transverse with respect to $\{\mu\},\{\kappa\},\{\rho\}$ if the index set $\{\kappa\}$ is nonempty and they are transverse with respect to $\{\mu\},\{\lambda\},\{\tau\}$ if the index set $\{\lambda\}$ is nonempty;

(iii) $\phi^{\{\mu\}\{\nu\}\{\kappa\}\{\lambda\}\{\rho\}\{\tau\}}_{\{\alpha\}\{\beta\}}(p,\theta)$ are σ-traceless with respect to any pair of a Lorentz-vector and a Lorentz-spinor indices of the form $(\mu_i,\alpha_j),(\kappa_i,\alpha_j),(\rho_i,\alpha_j),(\mu_i,\beta_j)$, $(\lambda_i,\beta_j),(\tau_i,\beta_j)$. Recall that σ-tracelessness of an arbitrary spin-tensor $X^{\mu\cdots}{}_{\alpha\ldots}$ means $(\sigma_\mu)^{\alpha\alpha'}X^{\mu\cdots}{}_{\alpha'\ldots}=0$.

Next, we observe that harmonic superfields belonging to $\mathscr{H}_0$ in fact depend on the Lorentz-spinor harmonics $v^{\pm\frac{1}{2}}_\alpha$ not in an arbitrary way but only through the light-like composites $u^{\pm}_\mu = v^{\pm\frac{1}{2}}\sigma_\mu v^{\pm\frac{1}{2}}$ (see (3.36), (2.11)). Therefore the field from $\mathscr{H}_0$ depend only on 45 independent combinations of the harmonic variables u^a_μ, u^+_μ (accounting for their kinematical constraints), which implies that the general solution of the 45 Dirac constraint equations

$$(D^{ab},D^{-+},D^{+a},D^{-a})\phi(z)=0, \qquad (\text{on } \mathscr{H}_0 \ (3.36),(2.11)) \qquad \text{(C.8)}$$

is $\phi(z)=\phi(x,\theta)$, i.e. it is constant with respect to (u,v) (cf. (2.19)).

The analog of the Dirac system (C.8) for the more general harmonic superfields (C.6) with external SO(8) × SO(1,1) indices reads:

$$(\hat{D}^{ab},\hat{D}^{-+},D^{+a},\hat{D}^{-a})\phi(z)=0, \qquad \text{(C.9)}$$

where $\hat{D}^{ab}$, $\hat{D}^{-+}$ have "spin" parts Σ^{ab}, $\hat{q}$ as in (C.4), (C.5) and $\hat{D}^{-a}$ accordingly reads (3.29):

$$\hat{D}^{-a}\equiv D^{-a}-\frac{\partial^a}{\partial^+}\hat{q}-\frac{\partial_b}{\partial^+}\Sigma^{ab}. \qquad \text{(C.10)}$$

Let us point out that the system (C.9) is consistent only on-shell, i.e. when $(-\partial^2)\phi(z)=0$, since (recall eq. (4.2)):

$$[\hat{D}^{-a},\hat{D}^{-b}]=(\partial^+)^{-2}\Sigma^{ab}(-\partial^2).$$

Now, inserting the general expression (C.6) into the system (C.9) and using the fact that the D's annihilate the SO(8) γ's [10] we get precisely eqs. (C.8) for the coefficient harmonic superfield $\phi^{\{\mu\}\{\nu\}}_{\{\alpha\}\{\beta\}}(z)$. Therefore, the general solution of eqs. (C.9) read:

$$\phi(z)\big|_{\text{on-shell}} = u^{\pm}_{\mu_1}\ldots u^{\pm}_{\mu_q}u^{c_1}_{\nu_1}\ldots u^{c_l}_{\nu_l}$$
$$\times\left(v^{+\frac{1}{2}}\sigma^{d_1}\right)^{\alpha_1}\ldots\left(v^{+\frac{1}{2}}\sigma^{d_m}\right)^{\alpha_m}\left(v^{-\frac{1}{2}}\sigma^{e_1}\right)^{\beta_1}\ldots\left(v^{-\frac{1}{2}}\sigma^{e_n}\right)^{\beta_n}\phi^{\{\mu\}\{\nu\}}_{\{\alpha\}\{\beta\}}(x,\theta). \tag{C.11}$$

Eq. (C.11) is the precise statement of the on-shell *pure-gauge* nature of the auxiliary harmonic variables (u, v) (2.4) for arbitrary harmonic superfields carrying external $\mathrm{SO}(8)\times\mathrm{SO}(1,1)$ indices. Namely, on-shell, the whole dependence of $\phi(z)$ on (u, v) is only through a *fixed* monomial in (u, v) carrying the external $\mathrm{SO}(8)\times\mathrm{SO}(1,1)$ indices of $\phi(z)$ whereas the physical fields are contained in the ordinary superfield $\phi^{\{\mu\}\{\nu\}}_{\{\alpha\}\{\beta\}}(x,\theta)$.

Property (C.11) exactly parallels analogous properties of $D=4$ harmonic superfields in [20].

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